## 2.2 Existence Statements and Negation

The negation of a generalization is an existence statement. Also, the negation of an existence statement is a generalization. In this section we illuminate these relationships by first inspecting the meaning and usage of the word *all*.

*All* is a word in both English and Mathematics. We all know what it means, but that does not mean that it is always used properly. Consider the following assertion: "All the letters in the string are a's."

String 1: *aaaaaaaa* "All the letters are *a*'s."

String 2: *aabaabaa* "Not all the letters are *a*'s."
"Some are not *a*'s."
But, "All the letters are not *a*'s" is wrong.

About String 2 we cannot say "All are not *a*'s." There is an important difference between String 2 and String 3.

String 3: *deffdijj* "All the letters are not *a*'s."

In String 3, all the letters are not *a*'s. **All** of them.

In the sentence, "All the letters are not *a*'s", the *all* refers to the letters, and the *not* refers to *a*. "All the letters are not *a*'s" correctly describes String 3. It is false about String 2, *aabaabaa*.

If "all are not," then "some are not" and "not all are."

String 4: *kkgggttt* "All are not *a*'s."
"Some are not *a*'s."
"Not all are *a*'s."

"All are not" says more than "some are not." If "All are not *a*'s" then "Some are not *a*'s." String 2, *aabaabaa*, shows the converse is false.

In Mathematics, the term *there exists* is used to negate a generalization using *all*. *There exists* corresponds to *there is* in English. "In String 2, *aabaabaa*, there exists a letter that is not an *a*."

**There is another possibility besides "all are" and "all are not."**
**It may be that "there exist" some.**

Don't let the "a" in "a letter" fool you. In Mathematics,

**"There exists a ..." means**
**"There exists at least one ...".**
**The singular form in Mathematics does not exclude the plural,**
**and the plural does not exclude the singular.**

**Some**.
String 3: *deffdijj* "Some letters are not *a*'s"
"There exists a letter which is not an *a*."
"Some letters are *e*'s."

In English, when we say *some*, we usually do not mean *all*. Otherwise we would have said *all*. To say "Some letters are not $a$'s" of String 3 would be regarded as true, but not the whole truth. But that is English. In Mathematics that usage of *some* is acceptable. The use of *there exists* is slightly preferable: "There exists a letter in String 3 which is not an $a$." This is obviously true, and should not be misleading.

About String 3, *deffdijj*, would you be willing to say, "Some letters are $e$'s"? Well, one is. And in Mathematics, one is enough. The plural, $e$'s, does not demand there be two or more. Yes, some are $e$'s. There exists an $e$.

**Definition 1.** *Let $x$ represent any variable and $S(x)$ represent an open sentence with that variable. A sentence explicitly or implicitly of the form "There exists $x$ such that $S(x)$" is called an* ***existence statement****.*

Example 1: "There exists $x$ such that $x^2 > 25$."
"$x^2 > 25$ for some $x$."

These are two equivalent true existence statements. $x = 10$ proves it true, because then $x^2 = 100 > 25$. To prove existence, a single example suffices.

*Such that* has the meaning of *and* in existence statements.

"There exists an $x$ **and** $x^2 > 25$"

would have the same meaning, but *such that* seems clearer. ◇

**Advice 2** (**Proving Existence**)**.** *One common way to prove an existence statement is to*

1) *Exhibit a candidate [for the thing that is asserted to exist], and*
2) *Then prove it has the properties it is claimed to have.*

Example 2: Conjecture: Let $y > 0$. There exists $x > 0$ such that $x < y$.

Proof: Let $y > 0$.
[Step 1] Choose $x = y/2$. [This is the candidate. Now show $x > 0$ and $x < y$.]
[Step 2] Because $y > 0$, $x = y/2 > 0$. [This verifies one property, $x > 0$.]
[Step 3] Because $y > 0$, $x = y/2 < y$. [This verifies the second property, $x < y$.]
◇

**The Negation of a Generalization**. The negation of a generalization is an existence statement. Axiom 3 relates the meanings of *all*, *not*, and *there exists*.

**Axiom 3** (**Negation of a Generalization**)**.** *Let $x$ represent any variable and $S(x)$ represent an open sentence with that variable.*

**3A:** *The negation of "For all $x, S(x)$" is logically equivalent to "There exists an $x$ such that not[$S(x)$]."*

**3B:** *The negation of "For all $x$ in $T, S(x)$" is logically equivalent to "There exists $x$ in $T$ such that not[$S(x)$]."*

**The negation of a generalization is an existence statement.**

Example 3: The negation of "All students are male" is "**There exists** a student who is **not** male" or "There exists a female student."

The negation of "All students **in this class** are male" is "There exists a student **in this class** who is **not** male," or "There exists a student in this class who is female." ◇

To disprove the sentence "All students in this class are male," all you have to do is find **one** female in the class. **One** exception **proves** a generalization is false.

Example 4: State the negation of "For all $x, f(x) \leq 18$."
Its negation is, "There exists $x$ such that $f(x) > 18$." ◇

Example 5: State the negation of "For all $x < 7, |x| \leq 7$."
Its negation is, "There exists $x < 7$ such that $|x| > 7$."

The hypothesis is retained, not negated. The $x$ which exists must still be less than 7.

-9 is an example of an $x$ which is less than 7 such that $|-9| > 7$.

The word *not* is no longer visible because "not($|x| \leq 7$)" was rewritten "$|x| > 7$." ◇

**Definition 4.** *The negation of a sentence is expressed in **positive form** if it is expressed without the word* not *(or, if that is not possible, with the word* not *moved inside the sentence as far as possible).*

**Definition 5.** *A **counterexample** to the generalization, "For all $x, S(x)$," is any example such that not[$S(x)$]. A generalization is false when there is a counterexample.*

Example 6: Conjecture: $2x \geq x$.

This is false. $x = -1$ is a counterexample. Then the conjecture says "$-2 \geq -1$" which is false. This proves that "$2x \geq x$" is a false generalization. There exists $x$ such that not($2x \geq x$). One counterexample proves a generalization is false. ◇

**Theorem 6** (**Negation of a Conditional Sentence**).
*The negation of "For all $x, H(x) \Rightarrow C(x)$"*
*is logically equivalent to "There exists an $x$ such that $H(x)$ and not[$C(x)$]."*

Example 7: Disprove: "If $-3 < x \leq 2$, then $|x| \leq 2$."

Proof that it is false: Let $x = -2.5$. Then $-3 < x \leq 2$ and $|x| > 2$. □

The object of the proof is to give an example where the hypothesis is true and the conclusion is not. Other examples are possible, but one example does it. ◇

Example 8: Theorem: If, for all $\epsilon > 0, |x| < \epsilon$, then $x = 0$.

Proof (by contrapositive): Suppose $x \neq 0$. Choose $\epsilon = |x| > 0$. Then $|x| \geq \epsilon$. □

The proof begins with the negation of "$x = 0$" and shows "not(for all $\epsilon > 0, |x| < \epsilon$)" which is equivalent to "There exists $\epsilon > 0$ such that $|x| \geq \epsilon$," by Axiom 3B. ◇

**Proving a Conditional False**. Suppose "$H(x) \Rightarrow C(x)$" is false. How could we prove that? We could show its negation is true.

Showing

"For all $x$, $H(x) \Rightarrow C(x)$" is false

would be the same as showing

"There exists $x$ such that $H(x)$ and not[$C(x)$]" is true,

which is the same as showing

"There exists $x$ such that $H(x)$ is true and $C(x)$ is false."

**Theorem 7.** *An example of $x$ such that $H(x)$ is true and $C(x)$ is false is a **counterexample** to the generalization "For all $x$, $H(x) \Rightarrow C(x)$."*

Example 9: Is the sentence, "$x < 4 \Rightarrow x^2 < 16$," true or false?

It is false.

Proof: $x = -10$ is a counterexample. "$-10 < 4$" is true and "$(-10)^2 < 16$" is false. ◇

One example suffices. One counterexample **proves** that a conditional sentence is false. A proof that an generalization is true may be long and involved, but a proof that a generalization is false is often simple.

**To prove a generalization false,**
**one counterexample suffices.**

Example 10: Conjecture: "If $b < c$, then $|b| < |c|$."

It is false.

Proof (that it is false): Let $b = -1$ and $c = 0$. Then $-1 < 0$, but $|-1|$ is not less than $|0|$. □

This is proof by counterexample because, in that case exhibited, the hypothesis is true and the conclusion is false. Because there are two variables, the counterexample exhibits both. ◇

**Subscripts**. To say "There exists $x$ such that $S(x)$" implies there exists at least one $x$. Sometimes, to emphasize the fact only one is guaranteed to exist, we use a subscript on $x$ to particularize it and make it clear the sentence is no longer a generalization. An equivalent sentence would be "There exists $x_0$ such that $S(x_0)$."

Example 11: Negate "$f(x) > 0$ for all $x$."

In positive form the negation is, "There exists an $x_0$ such that $f(x_0) \leq 0$."

The subscript is not at all necessary, but may help the reader. ◇

__Warning__. The negation of "For all $x$, $f(x) > 0$" is **not** "For all $x$, $f(x) \leq 0$." The negation of "For all $x$, $S(x)$" is **not** given by "For all $x$, not$[S(x)]$." Be very careful to avoid this common error.

**The negation of a generalization is *not* a generalization.**
**It is an existence statement.**

Example 12: Suppose "$x < 7 \Rightarrow f(x) \leq 45$" is false. What is true?

Do not say, "$x < 7 \Rightarrow f(x) > 45$." To be false, the original only requires **one** exception. It does not require **all** $x < 7$ to be exceptions.

The original conditional has the form "$H \Rightarrow C$" which has negation "$H$ and not $C$." Nevertheless, do not answer "$x < 7$ and $f(x) > 45$," which incorrectly omits the quantifier. You **must** mention "there exists" (or some synonym for it.) The negation is:

"**There exists** $x$ such that $x < 7$ and $f(x) > 45$."

Or, "There exists $x < 7$ such that $f(x) > 45$." ◇

Remember that conditionals with variables often have hidden universal quantifiers, so explicit mention of "for all" is often omitted. However, the existential quantifier may not be omitted. If you mean "there exists," do not omit it.

**Never omit "there exists"** (or some synonym for it).

**The Negation of Existence Statements**. By Theorem 1.3.16 on double negation, because the negation of a generalization is an existence statement,

**The negation of an existence statement**
**is a generalization.**

**Theorem 8** (**Negation of Existence Statements**).

**A:** *The negation of "There exists an $x$ such that $S(x)$," is*
*"For all $x$, not($S(x)$)."*

**B:** *The negation of "There exists $x$ in $T$ such that $S(x)$" is*
*"For all $x$ in $T$, not($S(x)$)."*

Example 13: The negation of "There exists an $x$ such that $f(x) > 0$,"
is "For all $x$, not$[f(x) > 0]$",
which can be rephrased in positive form as
"For all $x$, $f(x) \leq 0$." ◇

Example 14: "There exists $x$ such that $x > 5$ and $x^2 < 10$" is an existence statement.

It is false. To prove an existence statement is false we prove its negation is true. Its negation is a generalization.

Proof the original statement is false: If $x > 5$, then $x^2 > 25 \geq 10$. ◇

**Definitions as Existence Statements**. Many mathematical terms have definitions which are existence statements.

Example 15: Define *even number*.

We all know what it means for an integer to be even. The problem is to convert our knowledge into a proper mathematical definition. Because proofs consist of steps which are sentences, it is most helpful to define terms in complete sentences.

Definition A: $n$ is **even** iff there exists $j$ such that $n = 2j$. (The universal set is integers and "$n$" and "$j$" refer to integers.) Integers that are not even are said to be **odd**. ◇

Theorem B: If $n$ is even, then $n^2$ is even.

Proof: If $n$ is even, there exists $j$ such that $n = 2j$ [by definition, in the "$\Rightarrow$" direction]. Then $n^2 = (2j)^2 = 4j^2 = 2(2j^2)$, which is even [by definition, in the "$\Leftarrow$" direction, where "$2j^2$" here plays the role of "$j$" in the definition]. □

The proof showed the existence of a number (integer) such that $n^2$ is two times that number. The definition called it $j$ but it turned out to be $2j^2$ in the proof. That is fine because, in the definition, $j$ was a placeholder. (Technically, we are also using the fact that, if $j$ is an integer, so is $2j^2$, which follows from Closure, Result 6.1.0.

Theorem C: If $n^2$ is odd, then $n$ is odd.

Proof: Assuming we have as prior that numbers which are not even are odd, this is the contrapositive of the previous theorem. [No further proof is required.] □

Theorem D: If $n^2$ is even, then $n$ is even.

This is the converse of Theorem A. It requires a separate proof. Assume we have the next results as prior.

Result E: A number is odd iff it is $2k + 1$ for some $k$.

Result E is an existence statement. Using it, we can prove Theorem D by contrapositive and get two results for the price of one.

Theorem F: If $n$ is odd, then $n^2$ is odd.

This is the contrapositive of Theorem D, simplified by Definition A. The next proof proves both Theorems F and D.

Proof: Let $n$ be odd. Then there exists $k$ such that $n = 2k + 1$ by Result E, "$\Rightarrow$". Then

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

which is odd by Result E, "$\Leftarrow$". □

**The Syntax of Nested Quantifiers**. Syntax is that part of grammar that concerns how word order affects meaning. Generalizations and existence statements are often *nested*—combined one after the other—and the order matters a great deal.

**Convention 9.** *The order "For all $x$, there exists $y \ldots$" implies that the $y$ may depend upon the choice of $x$. The order "There exists $y$ such that for all $x \ldots$" implies that the $y$ does not depend upon the choice of $x$.*

Example 16: "For all $x > 0$, there exists $y > 0$ such that $y > x$."
It is true.
Proof: Let $x > 0$. Choose $y = x + 1$. Then $y > 0$ and $y > x$. □

This does not assert that a single $y$ works for all $x$. Perhaps this would be clearer if we said "**For each** $x \in S$, there exists $y$ ..." instead of "**For all** $x \in S$, there exists $y$ ...". *For each* and *For all* are synonyms, but *for each* is a better English version of what we mean. **Each** $x$ has an associated $y$. We do not mean that **all** $x$ have the same $y$. ◇

The other order means something else.

Example 16: [continued] "There exists $y > 0$ such that for all $x > 0, y > x$."
This has the same components with the order reversed.
The other one was true, but this is false! Order Matters!
Because $y$ is mentioned first, the syntax implies that there exists one $y$ that works for all $x$. This is not so. There is no single number larger than all numbers. Each number has a larger number. ◇

Example 17: True or false?
a) "There exists $y < 3$ such that for all $x < 3, x < y$."
b) "For all $x < 3$, there exists $y < 3$ such that $x < y$."
c) "For all $x \leq 3$ there exists $y \leq 3$ such that $x < y$."
d) "There exists $y \leq 3$ such that for all $x \leq 3, x < y$."

Order matters! Part (a) is false. There is no single $y$ that works for all $x$.
Part (b) is true. It really means "For each $x$." Each $x$ is allowed its own $y$ .
Proof of (b): Let $x < 3$. Choose $y = (x + 3)/2$.
Claim: Then $y < 3$ and $x < y$. ◇

**Definition 10.** *In this text and your work, the word **claim** may be used to label a result that is lower-level than the main idea of the proof. Claims can be proved, and should not be asserted unless they are easy to prove. Your instructor will decide if any claims you make should have been proven instead of merely asserted.*

Example 17, part (c), "For all $x \leq 3$ there exists $y \leq 3$ such that $x < y$," is false.
Proof: Let $x = 3 \leq 3$. There is no $y \leq 3$ such that $3 = x < y$. □
Part (d) is also false. The negation of

"There exists $y \le 3$ such that for all $x \le 3$, $x < y$."
is "For all $y \le 3$, there exists $x \le 3$ such that $y \le x$."
This negation is true so (d) is false.
Proof: Let $y \le 3$. Choose $x = 3$. [or, choose $x = y$. There is more than one way to do this proof.] Then $x \le 3$ and $y \le x$. □

Word order is extremely important in mathematics, but somewhat less so in English.

Example 18: Consider these combinations of a generalization and an existence statement.
"All students have an ID number."
"There is an ID number for all students."
The second is written as if there were **one** ID number, the same for all students. But that is obviously wrong, so we instantly reject that interpretation and decide it means "Each student has an ID number." In English we often know what is intended, even if something else is said. ◇

**Advanced Negations**. In the process of negation, $\forall$ changes to $\exists$ and vice versa, and the inside is negated. First write or rewrite the original sentence in left-to-right form with quantifiers preceding the letters they quantify. Then the negation simply reverses the quantifiers and puts the *not* on the rightmost sentence. Look for that pattern in this theorem.

**Theorem 11.** *Let $S(x, y)$ be an open sentence with variables $x$ and $y$, and $R$ and $T$ be sets.*
**11A:** *not [$\forall x\ \exists y\ S(x, y)$] is logically equivalent to*
$\exists x\ \forall y$ *not*$[S(x, y)]$.
**11B:** *not [$\exists x\ \forall y\ S(x, y)$] is logically equivalent to*
$\forall x\ \exists y$ *not*$[S(x, y)]$.
**11C:** *not [$\forall x \in R,\ \exists y \in T,\ S(x, y)$] is logically equivalent to*
$\exists x \in R,\ \forall y \in T,$ *not[$S(x, y)$].*
**11D:** *not [$\exists x \in R,\ \forall y \in T,\ S(x, y)$] is logically equivalent to*
$\forall x \in R, \exists y \in T,$ *not[$S(x, y)$].*

Example 17, part (a), revisited:
Conjecture (a): "There exists $y < 3$ such that for all $x < 3, x < y$."
Symbolically, it says, "$\exists y < 3, \forall x < 3, x < y$," or "$\exists y < 3 \ni \forall x < 3, x < y$."
The negation is "$\forall y < 3, \exists x < 3, x \ge y$."
The conjecture is false because the negation is true.
Proof: Let $y < 3$. Choose $x = y$. Then $x < 3$ and $x \ge y$. □

**Comment 12** (**Existence Proofs**). *Again, the primary way to prove existence is to*
1) *Exhibit a candidate (for the thing which is asserted to exist), and*
2) *Verify the candidate has the properties it is claimed to have.*

The proof above exhibited $x$ (which is claimed to exist) as $y$. The syntax says $x$ may depend upon $y$. Then the proof continued on to show that $x$ had the desired properties.

Example 19: Theorem: If, for all $\epsilon > 0, c < d + \epsilon$, then $c \leq d$.

The proof is by contrapositive. The contrapositive is

"If $c > d$, then not(for all $\epsilon > 0, c < d + \epsilon$)"

which is equivalent to

"If $c > d$, then there exists $\epsilon > 0$ such that $c \geq d + \epsilon$."

Proof: Let $c > d$. Choose $\epsilon = c - d$. Then $\epsilon > 0$. Also, $c = d + (c - d) = d + \epsilon$, so $c \geq d + \epsilon$. ◇

Textbook proofs often do not mention the logic. The reader is supposed to be familiar with common proof reorganizations that employ negation (Section 1.4), such as the contrapositive.

Example 20: Give the negation of this conjecture: "If $a > 0$, then $ax^2+bx+c = 0$ has no real-valued solutions whenever $c > 0$."

This requires identifying the hypotheses and quantified variables in the original. *Whenever* is a synonym for *if*. Identifying the hypothesis, the original is: "If $a > 0$ and $c > 0$, then $ax^2 + bx + c = 0$ has no real-valued solutions."

Making the quantifiers explicit, it is:

"For all $a$, $b$, and $c$, if $a > 0$ and $c > 0$, then $ax^2 + bx + c = 0$ has no real-valued solutions."

Now it fits the arrangement which makes negation easy. The negation is: "There exist $a$, $b$, and $c$, such that $a > 0$, $c > 0$, and $ax^2 + bx + c = 0$ has at least one real-valued solution."

An example of this is a counterexample to the original conjecture. Choose $a = 1$, $c = 2$, and $b = -3$, for which the equation is $x^2 - 3x + 2 = 0$, which is equivalent to $(x - 1)(x - 2) = 0$, which has two real-valued solutions. ◇

Example 21: Prove this is false: "If $c \geq 0$, then $a \leq b$ iff $ca \leq cb$."

A counterexample is a proof. Do not just say "$c = 0$ is the reason." It is, but that is not a complete counterexample.

**A good counterexample assigns particular values to *all* the letters in the generalization.**

Let $c = 0$, $a = 3$ and $b = 2$. Then "$a \leq b$" is false but "$ca \leq cb$" is "$0 \leq 0$" which is true. They are not equivalent. The particular counterexample proves the generalization is false. ◇

Example 22: Give the negation of "No horizontal line intersects the graph of $f$ twice or more."

In this sentence, $f$ is given and not quantified. The negation is "There exists a horizontal line that intersects the graph of $f$ twice or more." ◇

**The negation of "none are" is "at least one is."**

Example 23: Prove this: Let $f(x) = 3x + 1$. There exists $d > 0$ such that if $x < 2 + d$, then $f(x) < 7.1$.

To prove this existence statement, exhibit a $d$ and show that it works.

Proof: Choose $d = .01$. [This is not the only possible choice.] Then $d > 0$.
Also, by hypothesis, $x < 2 + d$ so $x < 2.01$.
Then $3x < 3(2.01) = 6.03$.
Then $3x + 1 < 6.03 + 1 = 7.03 < 7.1$.
Substituting, $f(x) < 7.1$ [So this choice of $d$ yields the desired properties.] ◇

**Conclusion**. The negation of *all are* is **not** *all are not*. Avoid this common mistake. The negation of *all are* is *some are not* (that is, *there is at least one that is not*).

The negation of a generalization is an existence statement. Conditional sentences may be generalizations with the *For all* implicit. Their negations are existence statements.

The negation of "$H(x) \Rightarrow C(x)$" is **not** "$H(x) \Rightarrow$ *not*$[C(x)]$." It is "There exists $x$ such that $H(x)$ and not$[C(x)]$."

"For all" is often omitted. **Never omit "there exists"** (or some synonym for it).

Negations of complicated combinations of generalizations and existence statements can be obtained by patiently applying the theorems on negation or by using the automatic approach from symbolic logic.

Terms: Negation, existence statement, counterexample, positive form.

---

Exercises for Section 2.2, **Existence Statements and Negation**:

A1.* ☺ a) What type of sentence is the negation of a generalization?
b) What type of sentence is the negation of an existence statement?

A2.* ☺ State the negation of "$H(x) \Rightarrow C(x)$."

A3. ☺ For part (a), make four decisions. Determine which of these four sentences apply to the row. Do the same for (b) and (c).
1) all are $h$'s; 2) all are not $h$'s; 3) not all are $h$'s; 4) there exists an $h$.
a) *hhhhhhhhhhh* b) *hhhhhhyhhhh* c) *yyyyyhyyhyy*

A4. ☺ For part (a), make four decisions. Determine which of these four sentences apply to the row. Do the same for (b) and (c).
1) all are 2's; 2) all are not 2's; 3) not all are 2's; 4) there exists a 2.
a) 23888888999 b) 22222222222 c) 44448966555

—— ☺ Give the negation, in positive form, of

A5. For all $x, g(x) \leq 12$.

A6. For all $y, h(y) > 10$.

A7. If $|x| > 7$, then $x > 7$.

A8. $x \leq 3 \Rightarrow f(x) > 5$.

A9. If $x \in S$, then $x \leq 25$.

A10. For each $x$ in $T, x^2 = 25$.

A11. If $x > 42$, then $x$ is not in $S$.

A12. $x \leq 4$ when $x \in S$.

A13. For each $x, f(x) = g(x)$.
A14. For all $x \in [0,2], f(x) = 0$.
A15. $f(x) > g(x)$ when $x > 6$.
A16. $f(x) > x$ when $x \in (0,1)$.

—— ☺ Disprove
A17. $|x+1| > |x|$.
A18. $|x-1| < |x|$.
A19. $x < 5 \Rightarrow x^2 < 25$.
A20. $bc = 0 \Rightarrow c = 0$.
A21. $|x+1| - 1 = |x|$.
A22. $b < c$ and $c > 0 \Rightarrow |b| < |c|$.
A23. $x^2 > 100 \Rightarrow x > 10$.
A24. $5x > 3x$.
A25. $a + b \geq a$.
A26. If $x^2 = 9$, then $x = 3$.
A27. $x^2 - 2x + 1 > 0$ for all $x$.
A28. If $x^2 - 5x = 0$, then $x = 0$.

—— ☺ Correct the English:
A29. "All batteries are not alike."
A30. "All beers are not alike."

—— ☺ (A31-32) Suppose this is true: If $x > 3$, then $f(x) < 4$. Which of these follow logically?
A31. a) If $x \geq 3$, then $f(x) < 4$.
b) If $x > 3$, then $f(x) \leq 4$.
c) If $f(x) > 4$, then $x \leq 3$.
d) If $f(x) \geq 4$, then $x < 3$.
e) If $x > 4$, then $f(x) \leq 4$.
f) If $x > 2$, then $f(x) < 4$.
A32. a) If $f(x) > 3$, then $x < 4$.
b) If $f(x) > 5$, then $x < 4$.
c) $f(2) > 3$.
d) $f(7) \neq 12$.
e) If $f(x) \geq 4$, then $x < 5$.
f) If $f(x) = 6$, then $x \leq 2$.
g) If $f(x) = 3$, then $x \leq 3$.

—— ☺ (A33-34) Suppose this is true: If $x \leq 5$, then $f(x) > 7$. Which of these follow logically?
A33. a) If $x < 3$, then $f(x) > 7$.
b) If $x > 5$, then $f(x) \leq 7$.
c) If $f(x) > 4$, then $x > 5$.
d) If $f(x) \geq 8$, then $x < 6$.
e) If $x < 4$, then $f(x) > 7$.
f) If $x < 2$, then $f(x) > 4$.
A34. a) If $f(x) < 3$, then $x > 4$.
b) If $f(x) < 5$, then $x > 4$.
c) $f(2) > 3$.
d) $f(5) \neq 6$.
e) If $f(x) \geq 4$, then $x >5$.
f) If $f(x) = 8$, then $x \leq 2$.
g) If $f(x) = 7$, then $x > 3$.

---

B1.* "$A$" and "not $A$" cover all the possibilities. Do "all are ..." and "all are not ..." cover all the possibilities? Explain clearly.

B2. Give a new example of a particular property (and a universal set) where both of the sentences "All have that property," and "All do not have that property," are false.

B3. a) State, in positive form, the negation of "$ab > 0 \Rightarrow a > 0$ and $b > 0$."
b) Prove the original generalization false.

B4. Restate as an explicit existence statement: "The rational numbers are not a subset of the integers."

—— ☺ Decide if these conjectures are true. If true, just say so. If false, say so and include a counterexample.
B5. ☺ Conjecture: $bc > 25 \Rightarrow b > 5$ or $c >5$.
B6. ☺ Conjecture: If $x < z$, then $x^2 < z^2$.
B7. ☺ Conjecture: If $x > |b|$, then $x > b$.

B8. ☺ Conjecture: If $x > b$, then $x > |b|$.
B9. ☺ Conjecture: $x + |y| \leq | x + y|$.
B10. ☺ Conjecture: $b < c \Rightarrow |b| < |c|$.
B11. ☺ Conjecture: If $a < b$ and $c < d$, then $a - c < b - d$.
B12. ☺ Conjecture: $|x|$ =c iff $x = c$ or $x = -$c.
B13. ☺ Conjecture: $a = b$ iff $ca = cb$.
B14. Conjecture: $|3 - 2x| < 7 \Rightarrow 3 < 7 + 2x$.

— ☺ **Negation**. Give the negation, in positive form, of each sentence.
B15. [Let $S$ be given.] If $x \in S$, then $x > 5$.
B16. [Let $T$ be given.] $x \leq 12$ for all $x \in T$.
B17. [Let $S$ and T be given.] If $x \in S$, then $x \in T$.
B18. [Let $S$ be given.] If $x \in S$, then $|x| \leq 25$.
B19. No polynomial has exactly three local extrema.
B20. No track team member is an all-American.
B21. All basketball players scored at least four points.
B22. All increasing sequences have limits.
B23. [Let $f$ and $g$ be given] For all $x, f(x) \geq g(x)$.
B24. [Let $f$ and $b$ be given] For all $x, |f(x)| \leq b$.
B25. Every horizontal line intersects the graph at most once.
B26. A horizontal line intersects the graph (at least) twice.
B27. No vertical line intersects the graph twice (or more).
B28. A vertical line intersects the graph twice (or more).
B29. [About several piles of balls] At least one pile has at least two balls.
B30. [About several piles of chips] At most one pile has more than 50 chips.
B31. [About several piles of chips] No pile has more than 20 chips.
B32. [About several piles of chips] At least one pile has at most 15 chips.
B33. No horizontal line intersects the graph twice or more.
B34. If $P(x)$ is a polynomial of degree four, then it has three local extrema.
B35. [Let $S$ and T be given.] If $x \in S$ and $x > 7$, then $x \in T$.
B36. [Let $S$ and T be given.] If $x \in S$, then $x \in T$ and $x < 5$.

B37. [Let f be given] a) Give the negation, in positive form, of "If $x < z$, then $f(x) < f(z)$."
b) Prove: $f(x) = x^2$ satisfies the negation, and therefore does not satisfy the given condition.

B38. Let $x^2 + y^2 = 1$ create a set of ordered pairs, $(x, y)$. a) Give the negation, in positive form, of "If $x_1 = x_2$, then $y_1 = y_2$."
b) Prove the negation is true and therefore the original is false.

— **Syntax of Nested Quantifiers**
B39. ☺ Decide if the syntax permits $x$ to depend upon $y$ (Yes or no):
a) "For $y > 0$ there exists $x > 0$ such that $f(x) = y$."
b) "There exists $x > 0$ such that for all $y$, $f(x) = y$."

B40. ☺ Decide if the syntax permits $x$ to depend upon $y$ (Yes or no):
a) "For each $y > 0$ there exists $x > 0$ such that $x < y$."
b) "There exists $x > 0$ such that for all $y > 0$, $x < y$."

— **True or false?**
B41. ☺ True or false?
a) "There exists $y \in (0,8)$ such that for all $x \in (0,8)$, $x \le y$."
b) "For each $x \in (0,8)$ there exists $y \in (0,8)$ such that $x \le y$."

B42. ☺ True or false?
a) "There exists $y \in [0,5]$ such that for all $x \in [0,5], x \le y$."
b) "For each $x \in [0,5]$ there exists $y \in [0,5]$ such that $x \le y$."

B43. ☺ True or false?
a) "There exists $y \in (0,\infty)$ such that for all $x \in (0,\infty), x \ge y$."
b) "For each $x \in (0,\infty)$ there exists $y \in (0,\infty)$ such that $x \ge y$."

B44. ☺ True or false?
a) "There exists $x > 0$ such that for all $y > 0, x < y$."
b) "For each $y > 0$ there exists $x > 0$ such that $x < y$."

— **Negation**
B45. ☺ a) Give the negation in positive form of: "There exists $x$ such that for all $y, x \ge y$." b) The negation is true (and the original is false). Prove it.
B46. ☺ a) Give the negation in positive form of: "For each $b$, there exists $x > 1$ such that $f(x) > b$." b) If $f(x) = 1/x$, the negation is true. Prove it.
B47. ☺ a) Give the negation in positive form of: "For each $x \ge 0$, there exists $y \ge 0$ such that $y < x$." b) The negation is true. Prove it.
B48. ☺ a) Give the negation in positive form of: "There exists $y > 0$ such that for all $x > 0, y < x$." b) The negation is true. Prove it.

— ☺ **Disproof.** Determine if the given fact would be enough to disprove the given conjecture.
B49. Conjecture: For all $x, f(x) > 7$. Fact: $f(2) = 5$.
B50. Conjecture: For all $x, f(x) > 7$. Fact: $f(1) = 9$.
B51. Conjecture: For all $x, f(x) > 7$. Fact: There exists $x$ such that $f(x) \le 7$.
B52. Conjecture: For all $x > 4, f(x) > 7$. Fact: $f(3) = 5$.
B53. Conjecture: For all $x > 4, f(x) > 7$. Fact: $f(6) = 4$.
B54. Conjecture: For all $x > 4, f(x) > 7$. Fact: $f(6) = 9$.
B55. Conjecture: For all $x > 4, f(x) > 7$. Fact: There exists $x$ such that $f(x) \le 7$.
B56. Conjecture: For all $x > 4, f(x) > 7$. Fact: There exists $x$ such that $x < 4$ and $f(x) > 7$.

B57. ☺ Suppose this is true: If $x < 7$, then $f(x) \ge 10$. Which of the following follow logically (FL)?

a) $f(x) > 11$ if $x = 6$.
b) $x > 7$ whenever $f(x) < 9$.
c) If $x = 8$, then $f(x) > 9$.
d) If $f(x) = 9$, then $x \ne 6$.
e) $f(x) \ne 9$ when $x < 4$.

B58. ☺ Suppose this is true: If $x > 5$, then $f(x) < 9$. Which of the following follow logically (FL)?

a) If $x = 9$, then $f(x) < 10$.
b) If $f(x) = 10$, then $x \ne 5$.
c) $f(x) \ne 10$ when $x \ge 6$.
d) If $x = 7$, then $f(x) \ne 12$.
e) $x < 8$ whenever $f(x) > 12$.

— ☺ **Proof** Suppose we want to prove "$H(x) \Rightarrow x > 7$" for some hypothesis H. Would proving this suffice?

B59. $H(x) \Rightarrow x > 8$
B60. "$H(x) \Rightarrow x > 6$
B61. not$(H(x)) \Rightarrow x \leq 7$
B62. not$(H(x)) \Rightarrow x \leq 8$
B63. not$(H(x)) \Rightarrow x \leq 6$
B64. $x < 7 \Rightarrow$ not$(H(x))$
B65. $x \leq 7 \Rightarrow$ not$(H(x))$
B66. $x = 2 \Rightarrow$ not$(H(x))$

— **Negations from Calculus**. Give the negations in positive form.

B67. Suppose $f$ is given. Give the negation of "For $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(t)| < \epsilon$ whenever $|x - t| < \delta$ ."

B68. Suppose $f$ and $x_0$ and $L$ are given. Give the negation of "For all $\epsilon > 0$ there exists $\delta > 0$ such that, if $0 < |x - x_0| < \delta$, then $|f(x) - L| < \epsilon$."

B69. Suppose the sequence $\{a_n\}$ is given. Give the negation of "For $m$ there is $n^*$ such that $a_n > m$ whenever $n > n^*$.

B70. Suppose the sequence $\{a_n\}$ is given. Give the negation of "For $M > 0$ there exists $N$ such that $n > N$ implies $a_n > M$."

— ☺ **Nested quantifiers**.

For each conjecture, state if it is true or false. [If your instructor also wants a proof, the proof will be called "Part (b)" and you will be requested to do both Parts (a) and (b).]

B71. Conjecture: Let $S = [0, 1)$. For all $x \in S$ there exists $y \in S$ such that $x < y$.
B72. Conjecture: Let $S = [0, 1)$. There exists $y \in S$ such that for all $x \in S$, $x < y$.
B73. Conjecture: Let $S = [0, 1)$. For each $x \in S$, there exists $y \in S$ such that $y < x$.
B74. Conjecture: Let $S = [0, 1)$. There exists $y \in S$ such that $x > y$ for all $x \in S$.
B75. Conjecture: Let $S = [0, 1)$. For each $x \in S$ there exists $y \in S$ such that $y \leq x$.
B76. Conjecture: Let $S = [0, 1)$. There exists $y \in S$ such that $x \geq y$ if $x \in S$.
B77. Conjecture: Let $S = (0, \infty)$. If $x \in S$, then there exists $y \in S$ such that $y < x$.
B78. Conjecture: Let $S = (0, \infty)$. There exists $y \in S$ such that $y < x$ for all $x \in S$.
B79. Conjecture: Let $S = (0, \infty)$. If $x \in S$, then there exists $y \in S$ such that $y > x$.
B80. Conjecture: Let $S = (0, \infty)$. There exists $y \in S$ such that $y > x$ for all $x \in S$.
B81. Conjecture: If, for all $c > 0, x < b + c$, then $x < b$.
B82. Conjecture: If, for all $c > 0, |x| < c$, then $x = 0$.

B83. Give a simplified form of the negation of "$H \Rightarrow (B \wedge C)$."

B84. Resolve this conjecture: Integers divisible by $n$ and by $m$ are divisible by $nm$.

B85. ☺ The following conjecture could be interpreted two ways. Explain the possible confusion.
Conjecture: [For a given $f$] "$f(x) \geq 0$ or $f(x) < 0$."

B86. Criticize the argument: Conjecture: "For all $f, f(2) > f(0)$."
Argument: Suppose not. Then $f(2) \leq f(0)$. But, when $f(x) = x^2$, $f(2) = 4 > 0 = f(0)$, and the assumption is contradicted, so the result is true.

— **Negations**

B87. Conjecture: If $x < 1 + \epsilon$ for all $\epsilon > 0$, then $x < 1$. It is false.
a) Give a counterexample. b) Give the negation of the conjecture.

B88. Conjecture: If $x > 3 - \epsilon$ for all $\epsilon > 0$, then $x > 3$. It is false.
a) Give a counterexample. b) Give the negation of the conjecture.

B89. Conjecture: If $x < 1 + \epsilon$ for all $\epsilon > 0$, then $x \leq 1$. Give the contrapositive of the conjecture. [A proof would follow easily from the contrapositive.]

B90. Conjecture: If $x > 3 - \epsilon$ for all $\epsilon > 0$, then $x \geq 3$. Give the contrapositive of the conjecture. [A proof would follow easily from the contrapositive.]

B91. a) Give the negation of "If $c > 0$, then $ax^2 + bx + c > 0$."
b) Prove the original is false.

B92. Give the contrapositive of this [$b$ and $S$ are given]: "All upper bounds of $S$ are at least $b$."

---

C1. Define the concept of an integer being "divisible by 3."
C2. Suppose $y$ depends upon $x$. Define "$y$ is directly related to $x$."
C3. Suppose $y$ depends upon $x$. Define "$y$ is inversely related to $x$."

— Let $H$ denote some hypothesis. Let $f$, $g$, $R$, $S$, and $T$ be fixed. State the negation, in positive form, of
C4. $H \Rightarrow f = g$. C5. $H \Rightarrow S \subset T \cup R$.

C6. Here is sentence about primes: "$n > 1$ is not a prime number iff there exist $i$ and $j$ such that $n = ij$ and $i$ >1 and $j > 1$." Use negation (and still assume $n = ij$) to complete the sentence "$n$ is a prime number iff ...".

C7. Axiom 2.2.3A on negating a generalization is like a DeMorgan's Law. Which one? How?
C8. Theorem 2.2.3B on negating an existence statement is like a DeMorgan's Law. Which one? How?

— **Existence of functions**
C9. Prove: There exists a function, $f$, such that $f(x + 1) = f(x) + 2$, for all $x$.
C10. Prove: There exists a function, $f$, such that $f(x + 1) = f(x) - 3$, for all $x$.
C11. Prove: There exists a function, $f$, not identically zero, such that $f(x+1) = 2f(x)$, for all $x$.
C12. Prove: There exists a function, $f$, not identically zero, such that $f(2x) = 4f(x)$, for all $x$.
C13. Prove: There exists a function, $f$, not identically zero, such that $f(2x) = 8f(x)$, for all $x$.