Contents

Preface: To the Instructor

Goals. Students will learn the basic concepts, reasoning patterns, and language skills that are fundamental to higher mathematics. The skills include the ability to

- read with comprehension
- express mathematical thoughts clearly
- reason logically
- recognize and employ common patterns of mathematical thought
- read and write proofs

Audience. This text is designed for future mathematics teachers and mathematics majors.

This text can be used at either the sophomore or junior level. Ideally, the students would be sophomores who intend to continue in mathematics.

Approach. This text covers the entire language of mathematics including the uses of variables, the conventions of the language, reading theorems and definitions, logic, and all the other language features involved in proofs. The text does not assume that the students are already strong in logic or the underlying language of mathematics; rather, it aims to make them so. We believe that interested students can become good at reading and writing mathematics, including proofs, if they are taught logical thinking and the notations and patterns of mathematical writing.

The exercises are a critical part of the approach. Many are specially marked with ☺ and can be asked and answered aloud in class to give both the instructor and the students useful feedback. When it comes time to do proofs, many assertions in the exposition and homework are posed as "conjectures" to involve students in deciding if statements are true or false, as well as proving or disproving them.

A key idea is that the students should be doing most of the work. After the basics are thoroughly covered in Chapters 1 and 2, the remaining chapters provide numerous statements for students to prove or disprove. In Part 2, Practice, most of the straightforward proofs are left to the students. The topics in Chapters 4 through 9 are not treated as sequences of clever proofs for the students to study, rather as sequences of results and conjectures for the students to actively address. Some of the hard proofs and some sample proofs are given, but most of the work is left to the students.

Essentials. Part 1, "Theory of Proofs: Language and Logic" (Chapters 1 through 3), is essential. It constitutes a complete discussion of the symbolic language of mathematics, logic for proofs, and the theory of proofs.

If time remains, Part 2, Practice, provides many opportunities for the students to practice. At Montana State University the basic three-credit sophomore-level proof course covers Chapters 1 through 5.

Selections from the remaining chapters could constitute a successor course for future teachers.

Why Use This Text? This text includes the usual "transition to higher mathematics" topics. However, there are numerous reasons it is significantly different from other texts with the same goals.

This text

1. discusses the entire language of mathematics without assuming the students are already fluent.
2. has many good homework questions at the right level.
3. has many homework problems clearly marked (with ☺) as short-answer problems that are excellent for students to do aloud in class. They can be used to involve students and give immediate feedback to both the instructor and the students.
4. uses many conjectures to involve students in deciding if statements are true or false and to force them to become critical thinkers (e.g., conjectures pose the converse of a theorem, or something similar with a hypothesis omitted, or something that looks likely but is actually false).
5. has many valuable activities for students to do, even in class. It does not require each day to be a faculty lecture.
6. emphasizes precisely the logic and logical equivalences that are commonly used in mathematics. It explicitly shows how statements are commonly rearranged for proofs. It uses the logical forms of theorems to indicate how to organize proofs, especially, where to begin proofs and where to end them.
7. clarifies how formal definitions work, with many simple examples that illustrate how definitions are used in proofs.
8. clarifies all the facets to"translation," because two sentences can look different and say the same thing.
9. is clear about which prior results can be used in a proof by going deep enough into particular topics so students have a foundation to build on.
10. is clear, when writing proofs, about what part is the proof and which parts are instruction about how to do proofs. [Our instructional comments are distinguished by enclosing them within brackets.]
11. repeats good ideas and advice because students will not grasp everything the first time around.
12. has six chapters in Part 2 on various mathematical topics that provide practice. The results are carefully paced so the students have numerous activities at the right level of difficulty.

13. has an instructor's manual with comments on the sections and what you might emphasize and what your students might find difficult.

14. has a solutions manual.

Homework. ☺ The symbol "☺" marks problems that are very short and could be done aloud in class. Your students will rapidly find out if they are doing it right, and so will you.

Problems are classified as A, B, or C problems."A" problems should be straightforward. "B" problems reflect the current level of work and understanding expected of the students.

"C" problems are not required, usually because they are harder or because they address additional related concepts not discussed in the exposition and not used later in the text. However, C problems are interesting and may be assigned if you wish to go deeper into the material and the students are capable.

In-Class Work. In Part 1 there are many problems students can do aloud in class. They are marked ☺. They solidify important lessons, but are so short they will not take much time.

Parts of Chapter 3 and all of Chapters 4 through 9 are designed to be somewhat like mathematics research. Many results and conjectures in those sections are unproven or unresolved. Asking students to resolve them during class is an option.

Pace. The early material is straightforward. Each section in Chapters 1 and 2 can be covered in an about a class hour.

Chapters 3 through 9 are different, with sections that could take longer, or much longer, depending upon how much of the work the students are asked to do. For example, Section 3.2 has numerous results on absolute values that an instructor could prove rapidly. However, this text is designed to involve students. If the students are expected to resolve the conjectures by creating the proofs and disproofs themselves, such sections could take longer, or much longer.

Here is a possible pace. The number given is a number of 50-minute classes, including some student involvement. Each + sign refers to a potential additional day if students are expected to do the requested work, say, by putting homework on the board, working on conjectures in class, or responding in class to numerous questions.

Part 1:

CHAPTER 1: INTRODUCTION TO PROOFS

Section	Classes
1.1. Preview of Proof 1	1
1.2. Sets	1-2
1.3-1.6. Logic for Mathematics	4++

CHAPTER 2: SENTENCES WITH VARIABLES .	
2.1. Sentences with One Variable	1
2.2. Existence Statements and Negation	1
2.3. Reading Theorems and Definitions	1-2
2.4. Form and Deduction	1
2.5. Rational Numbers and Form	1+

CHAPTER 3: PROOFS	
3.1. Inequalities	1++
3.2 Absolute Values	2++
3.3. Theory of Proofs	1+
3.4. Proofs by Contradiction or Contrapositive	1+
3.5. Mathematical Induction	1++
3.6. Bad Proofs	1+

Part 2: At Montana State University and at Stonehill College in a three-credit sophomore-level course we cover Chapters 1 through 5. In a successor course, we resume with Chapter 6. However, It is not necessary to do Chapters 4 though 9 in the given order and it is not necessary to cover all the sections in any given chapter.

Chapter 4, Set Theory, is a natural next chapter

Chapter 5, Functions, does not require Chapter 4.

Chapter 6, Number Theory, has no prerequisites from Chapters 4 or 5.

Chapter 7, Group Theory, has prerequisites. It requires number theory through Section 6.4 and Sections 5.1 and 5.2.

Chapter 8, Topology requires set theory (Chapter 4) but nothing from the Number Theory or Group Theory chapters.

Chapter 9, Calculus, does not require Chapters 4 through 8, but the existence proofs in any of Chapters 4 through 8 would be helpful preparation for calculus. Chapter 9 could begin with either Section 9.1 on sequences or 9.2 on limits and derivatives.

Time is limited, so if you prefer breadth to depth you could easily and effectively omit the final sections of each chapter.

Organization. Section 1.1 is a preview of many of the key ideas of proof. [Relevant sections are cited in brackets.]

Section 1.2 is on sets because mathematics is a language and we need something to talk and write about. Sets are a good beginning topic for several reasons. The definitions of the terms (such as intersection and subset) use logical connectives, so they illustrate the logic we use. Also, for proofs, statements using set-theory terms are translated into equivalent symbolic and logical forms, so they illustrate the use of definitions and translation of recently-defined terms.

Sections 1.3 through 1.6 are on the logic of proofs with emphasis on precisely the logical reorganizations that are commonly used in proofs, such as the contrapositive.

Chapter 2 is on the uses of variables, generalizations, existence statements, how to read theorems and definitions, and how the forms of statements can be rearranged. For example, Figure 2.3.4 illustrates how to work with recently-defined terms.

Chapter 3 is on proofs in general, including representative-case proofs, existence proofs, proofs by contrapositive and contradiction, and proofs by induction. The first section, Inequalities, emphasizes that steps must be justified by prior results. Section 3.2 is on absolute values, and important topic in calculus. It begins to shift responsibility for precise thought from the instructor to the student. It has numerous conjectures, some true and some false, interspersed with theorems, in order to help students become critical thinkers. Sections 3.3 through 3.5 discuss the theory of proofs and proofs by contrapositive, contradiction, and induction. Section 3.6, Bad Proofs, requires students to judge arguments and recognize some of the most common types of errors.

Chapters 4 through 9 continue the discussion of proofs by providing practice.