Teaching about Inverse Functions

Warren Esty

Warren Esty is a professor of mathematics in the Department of Mathematical Sciences at Montana State University in Bozeman, Montana. He has published extensively in probability theory, statistics, and mathematics education. He has written two books, *The Language of Mathematics* and *Precalculus.* In his spare time he studies ancient Rome and Greece.
E-mail: westy@math.montana.edu

What should students know about inverse functions? The place to discover the value of inverse functions is not in the textbook sections that teach about inverses, but in the subsequent sections and courses where inverses are actually used. A review of the uses of inverse functions is employed here to evaluate the pedagogical value of various approaches to teaching about inverses.

Almost all textbook sections on inverses are similar. They address the idea that $f^{-1}(f(x)) = x$, the notation with "-1," the term "one-to-one," the use of horizontal-line test, and the concepts of *domain* and *range.* Each text teaches an algorithm for finding f^{-1} when (and only when) f is simple. The following review demonstrates that introductory textbook lessons and homework on inverses usually emphasize parts of the subject that do not reappear, and fail to emphasize other parts that appear frequently. This shows a need to rethink what is emphasized when inverses are taught. Analysis of the review suggests appropriate pedagogical changes.

Review of The Uses of Inverses

Consider the actual uses of inverses and the value of the concept. Inspection of precalculus and calculus texts shows that, after the introductory section on inverses, inverses appear primarily when equations need to be solved. For example, textbooks define and use the inverse sine function to solve equations of the form "$\sin x = c$." In this context students find two things difficult. One is the notation. The superscript "-1" in the notation for inverse sine, $\sin^{-1}$, looks like the notation for the reciprocal, which is a difficulty addressed by all texts. The other, more significant, difficulty is that there is a second triangle-angle solution to "$\sin x = c$" that is not obtained by the obvious calculator keystrokes. When an obtuse angle x satisfies "$\sin x = 0.98$," students often trust their calculators to find the solution and erroneously answer

"$x = 78.5°$" because they found $\sin^{-1} 0.98$ and never thought about the second solution. Students must learn that there is a second solution for triangles, "$x = 180° - \sin^{-1} 0.98$."

Texts also use inverse functions before trigonometry. For example, squaring is simply multiplication, but solving "$x^2 = c$" is much more complicated, so a name is given to the method of solution, the square-root function. Of special interest in this context is the additional complication that there is a second solution not given by the square-root function (and often forgotten by students). Similarly, in intermediate algebra and precalculus when monomials are discussed, the equation "$x^n = c$" is solved using the inverse function and there may or may not be complications, depending upon whether n is even or odd.

In precalculus, inverses also appear in the context of exponential functions. Logarithmic functions are needed to solve equations such as "$10^x = c$," "$e^{5x} = 12$," and "$10{,}000(1.04)^t = 15{,}000$." The graphs of $y = 10^x$ and $y = \log x$, as well as $y = e^x$ and $y = \ln x$, are mirror images of one another through the line $y = x$, because (b, a) is on the graph of the inverse when (a, b) is on the graph of a function, which is a typical lesson from the initial section on inverses.

In calculus, new uses of inverses are rare. Inverse trigonometric and inverse hyperbolic functions appear as integrals of certain algebraic functions, so the derivatives of these inverse functions must be obtained. The key prior result about inverses is $f(f^{-1}(x)) = x$. By differentiating this using the chain rule, the derivative of $f^{-1}(x)$ is derived, which yields the inverse function as the integral of its derivative. An infinite series for $\tan^{-1} x$ can be derived from its derivative and the sum of a geometric series.

Conspicuously absent in all intermediate algebra, precalculus, and calculus texts, after the initial section on inverses, is any occasion to derive $f^{-1}(x)$ for simple algebraic functions such as $f(x) = 3x + 1$. The algorithm for deriving f^{-1}, however conceptually significant it may be, is never used again.

Lessons and Inverses

This comprehensive review outlines what is important about inverses. It allows for a comparison of any exposition about inverses, and its emphasis on each concept, to the future value of the lesson. This article employs examples from only three popular texts, but the reader is encouraged to similarly evaluate the text she or he uses.

The review shows that the primary motivation for developing inverses is to name a method for solving equations that cannot be solved with simple techniques.

If students need to solve "$f(x) = c$" for a simple algebraic function such as $f(x) = 3x+1$, they can just solve the equation—there is no need to generalize the equation-solving process to find and name the inverse function. This is why inverses are never again derived (only defined). This also shows that the point of any lesson about how to derive f^{-1} from f must be conceptual, not computational. What concept is taught by deriving the inverse of $f(x) = 3x + 1$? It must be that f^{-1} generalizes the solution process and is used for solving equations.

Nevertheless, most precalculus texts fail to emphasize the primary context which motivates the author's interest in inverses— solving the equation "$f(x) = c$." For example, Dugopoloski (2002) introduces the section by saying "It is possible for one function to undo what another function does" and never mentions the equation-solving purpose. But the reason for the interest in *inverse sine* is so one can solve "$\sin x = c$." Somehow the typical exposition has lost track of the practical use of inverses.

Similarly, many texts teach an algorithm for finding f^{-1} that does not take advantage of the possibility of **perfectly** paralleling the process for solving "$f(x) = c$" for x. For example, Sullivan (2002) has students switch x and y before solving, which requires the students to solve for y, instead of solving for x as they would in the motivating problem. Look at this typical "switch letters first" exposition (Sullivan (2002), page 226).

"If f is defined by the equation

$$y = f(x)$$

then f inverse is defined by the equation

$$x = f(y)$$

The equation $x = f(y)$ defines f^{-1} *implicitly*. If this equation is solved for y, then the *explicit* form of f^{-1} is obtained, that is,

$$y = f^{-1}(x)."$$

True. But anyone looking at "$y = f(x)$" next to "$x = f(y)$" sees something is wrong. If the first is true, the second is false. So, the student must see the second as beginning a new and arbitrary algorithm: Use the same function, switch x to y and y to x. Then solve for y. The result is $f^{-1}(x)$. It works, but it's a mystery! The work will, unfortunately, not look like the work for solving "$f(x) = 17$," which it can and should in order to reinforce the concept that f^{-1} operates on images, y, of f and returns arguments, x, of f.

Why not use this algorithm? (1) Set $f(x) = y$. (2) Solve for x. The result is "$x = f^{-1}(y)$." Simple!

Teaching this is easy. Suppose $f(x) = 3x + 1$. To show that the process for solving "$3x + 1 = 17$" is the same as for "$3x + 1 = 92$," do both. Note that in each case you "Subtract 1 and then divide by 3." Students might observe that the number on the right does matter—different numbers yield different solutions. This observation enables us to focus attention on what students are really doing when they find the inverse function—they are abstracting the process from the numbers. One can say, "The number matters to the **solution**, but not to the solution **process**, which depends upon f and not upon c (or "y"). The solution process is a **function** (which is called f^{-1}), not a number." Teaching about inverses can help foster the algebraic habit of "abstracting from computation" (Driscoll 1999), but not if the x's and y's are arbitrarily switched first.

A "function-loop diagram" makes the relationship of f andf^{-1} clear (Figure 1). The function f takes x and yields y. The function f^{-1} takes y (perhaps 17) and returns x.

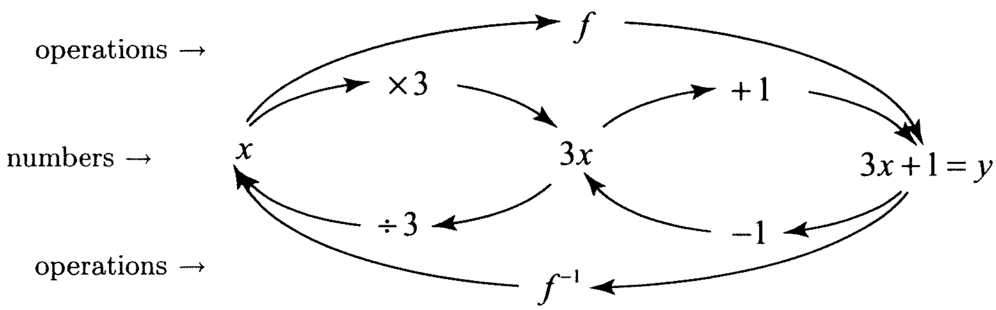

Figure 1: A function-loop. f takes x to y. f^{-1} takes y and returns x.

Of course, it is legal to switch letters. Although it is natural to think of f as operating on x and f^{-1} therefore operating on y (Figure 1), the letter used to define a function is not critical. The assertion "$f^{-1}(x) = (x - 1)/3$" has precisely the same information as "$f^{-1}(y) = (y - 1)/3$." However, deriving $f^{-1}(x)$ by switching letters *first* does not parallel the usual computation which being attempting to generalize. Deriving $f^{-1}(y)$ does, and students can switch letters *last*, if there is reason to. (However, often there is not. Giving $f^{-1}(y)$ is perfectly meaningful.) This can be a good occasion to explain again that functional notation describes a relationship between the argument and image, and the letters used to describe that relationship are not critical. It is a shame that this is usually the only context where letter-switching is emphasized, erroneously leading students to conclude that letter-switching has something to do with inverses, when it is really just a property of functional notation and letters could be switched in any context.

Switching letters before solving has another significant pedagogical disadvantage; it confuses the important distinction between argument and image. Some letters are used in a helpful manner to make this distinction—switching them would be wrong. The equation "$\sin\theta = c$" is fine, but the equation "$\sin^{-1}\theta = c$" is not. The symbol θ is fine for an angle, but not for the argument of inverse sine. Even if the letters are x and y, why would a teacher want to confuse which is which? When the problem states "$f(x) = y$," how can students be comfortable writing "$f(y) = x$"?

Texts mention that the domain and range switch too. This would be a good place to formulate *range* in relevant terms. The key idea is that *range* is essentially an equation-solving concept. Instead of only saying "The *range* of f is the set of all images of elements in the domain" one could state the equivalent, but truly meaningful, "The *range* of f is the set of all c such that the equation $f(x) = c$ has a solution."

Regardless of how the chosen algorithm for computing f^{-1} works, or which version of the algorithm works better, a review of texts shows that students will **never do this again**. Therefore, when the two approaches above are evaluated and compared, one is comparing how well they foster appropriate conceptual development. The key concepts are that f^{-1} is for solving equations, f^{-1} abstracts a process, and f^{-1} operates on the original y's to return the original x's. The method that solves for x (instead of switching letters and solving for y) is clearly preferable on all counts.

One-to-One

The review of the uses of inverses shows that the useful ones are special functions such as the square-root function and the inverse-sine function. Students have well-known difficulties remembering to think of the obtuse-angle solution to "$\sin\theta = c$." What can an introductory section on inverses do to help?

Sometimes (for some choices of f) there is more than one answer to "$f(x) = c$", and using a calculator's inverse function will find only one, so students must (1) think about whether there might be additional solutions, and (2) learn how to find them from the **one** given by the calculator. This is the point of the concept "one-to-one" and the horizontal-line test. They are used to separate out functions for which the equation "$f(x) = c$" might have more than one solution.

Unfortunately, most texts fail to make this point. Larson and Hostetler (2000) define "one-to-one" by stating that "$f(a) = f(b)$ implies $a = b$," but never hint that this might mean that if f is **not** one-to-one, then there might be more than one solution to "$f(x) = c$,"and those cases are tricky. Consequently, they do not note that, even if there is an inverse function defined (as there is when the equation is "$x^2 = c$" or "$\sin x = c$"), the inverse of c might not be the desired solution. And, if there is more than one solution, almost no texts take the occasion to mention how theorems are written using "or" to give the other solutions in addition to $f^{-1}(c)$ [For example, "$x^2 = c$ if and only if $x = \sqrt{c}$ or $x = -\sqrt{c}$"].

The concept *one-to-one* is also important for defining inverse functions. To obtain an inverse **function**, by the definition of *function*, there must be only one number returned. Does the definition of *one-to-one* explain this? The definition is either "If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$" (Sullivan), or its contrapositive, "If $f(b) = f(a)$, then $b = a$" (Larson and Hostetler). These formulations are used in proofs—but proofs are not the focus here. Which is easier for students to understand? Often neither is at all clear until they are illuminated with the horizontal-line test, "Every horizontal line intersects the graph of the function at most once" (Larson and Hostetler). Students who have used graphing calculators will be familiar with the idea that the intersection of two graphs yields a solution to an equation. The definition could be reformulated to take advantage of what students know about graphs: "The equation '$f(x) = c$' has at most one solution," or "If the equation '$f(x) = c$' has a solution, it has only one." Because solutions to the equation "$f(x) = c$" are found where the horizontal line $y = c$ intersects the graph of $y = f(x)$, these alternative definitions

(1) incorporate the horizontal-line test,
(2) fit perfectly with the definition of *range* that emphasizes solutions to "$f(x) = c$," and, most importantly for this context (which is not the study of abstract proofs),
(3) directly relate the concept of *one-to-one* to solving equations.

These alternative versions develop a concept that helps students avoid mistakes when solving equations. It immediately follows that a function is **not** one-to-one if and only if some value of c yields an equation "$f(x) = c$" with more than one solution. Therefore, if the function is **not** one-to-one, even if there is a nominal inverse, it will not necessarily find **all** the solutions. This is the important way to regard *one-to-one* because this addresses the mistake students make. The sine function has a nominal inverse, $\sin^{-1}$, but it does not find all the solutions to "$\sin x = 0.4$." All teachers have seen students overlook the second-quadrant solution. Here is also a good opportunity to discuss how theorems are written with "or" to express additional solutions. The fourth-power function has a nominal inverse, but it does not find all the solutions to "$x^4 = 37$." In contrast, the exponential function e^x has a nominal inverse and the inverse does find all the solutions because the exponential function is one-to-one.

Conclusion

A review of the uses of inverses reveals that the algorithm for computing f^{-1} is never used again after the introductory section. Inverses are most frequently encountered as methods of solving equations of the form "$f(x) = c$," and the most difficult lesson is that equations may have more than one solution when f is **not** one-to-one. Precalculus textbooks tend to emphasize the algorithm and rarely even mention the complications of the important case when f is not one-to-one, which suggests that textbook priorities have been misplaced.

The value of learning and practicing the algorithm is not computational, but could be conceptual. The method for finding f^{-1} has its natural interpretation in the context of solving "$f(x) = y$" for x. Similarly, "one-to-one," the horizontal-line test, "domain" and "range" have natural interpretations in that context. However, teaching the version which first switches x and y does not contribute to proper conceptual development.

Some might argue that, to find f^{-1}, first switching x for y to find $f^{-1}(x)$ "works" and the students can do it. Nevertheless, one should not judge learning based on whether an unimportant algorithm can be memorized. Teachers should prefer consistent notation to inconsistent notation, emphasize the actual context (solving equations), emphasize the real significance of the terms *one-to-one* and *range*, and provide a perfect parallel to the usual process of solving equations for x. Textbook expositions should emphasize the equation-solving context and discuss the possibility of solutions to "$f(x) = c$" other than the one returned by a calculator, $x = f^{-1}(c)$, because, in the future, solving equations (and, all too often, omitting an important solution) is most of what students will do with inverses.

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