Revised, Feb. 2015 to be more helpful, especially at the beginning of the course. (Pages 2-6 below.)

Instructor's Manual

for

Proof: Introduction to Higher Mathematics

Seventh Edition

by Warren W. Esty and Norah C. Esty

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Why Read This Manual?

This manual is intended to teach you, the instructor, some of what you would know about the course if you had already taught the course. Parts of this course will seem different to you, and your students have never taken another math course anything like this one. The class will go very well if you understand why we do what we do. Please take our advice and read some of this manual.

This manual also

1) suggests things to for you do in class,

2) suggests things for you to ask students to do in class,
   2A) (From the beginning) short-answer questions that clear up misconceptions, e.g. Section 1.2, most "A" problems.
   2B) (Later in the semester) longer questions for students to work in groups, possibly at the board

3) alerts you to various types of errors that we have seen that you might not expect students would make. E.g. in logic, many students reverse the truth values of "$x > 4 \Rightarrow x \geq 4$" and "$x \geq 4 \Rightarrow x > 4$." Who would have thought?

4) discusses the value of using our terminology (Some terms we use might not be familiar to you and you might wonder why we bring them up. Rest assured they are helpful and you will grow to like them, appreciate them, and use them. They really do help students conceptualize what is going on. We do not emphasize any terms that are valuable.)

5) identifies resources, such as some on-line homework
   Problems marked with a superscript W (for "WeBWork," the MAA program), such as 1.2, A19, have corresponding on-line problems both in the MAA on-line library and at the Montana State site (which you and your students are free to use without a password):
   http://webwork.math.montana.edu/webwork2/M242/

6) has things you can say in class that are useful
   "It is a goal of this course that you learn to read math well enough to learn math by reading it. How do you learn to read? You learn to read by reading! Read the text!"
   "In logic we study form, not meaning. Logic has nothing to do with meaning."
   "Logical equivalences allow you to replace one sentence with another. If you prefer the alternative form, use it."
   "The connective "if..., then..." is a **thing**. It has properties. A logical equivalence written with "A => B" is not about A or B, which are merely

placeholders, it is about 'if..., then...'"
"Identities (such as "$2x + 3x = 5x$) can be **written** with 'x', but they are not **about** x, they are about alternative operations and order."
"Tell me what to write." Sometimes a student response in class suggests the student knows something relevant, but has not phrased it correctly. If you say it correctly, you may be deriving the student of the chance to learn to say something precise. "Tell me what to write" gives them that chance.

General Recommendations.

1) Encourage your students to actually **read the text**. To this end, some professors assign some homework from the next section before covering it. One professor who uses the text assigns all the required homework before covering the section in class. However you choose to do it, make sure students realize that they are studying a written language that they must learn to read and write.

2) **Consider asking smiley-face questions aloud in class**. The text has numerous exercises designed to be done in class. Everyone learns from their responses and your feedback. You could start (or end) almost every day by having students do these types of problems. The exercises are easy to find on the fly because they are marked with a smiley face, ☺. For example, in Section 1.2, problems A33-A69 determine if letters and symbols are being used properly. Problems 1.2.A9ff give you a chance to see if they can pronounce things properly.

3) We recommend you **reserve a classroom with a lot of board space** or have a device for projecting student work. Beginning in Chapter 3 we frequently have students work in pairs or groups. By sending most or all of the students to the board simultaneously you can create a real collaborative research experience.

4) **Trust us about the value of using the terms we introduce** such as "open sentence" (Definition 1.3.1), "sentence-form definition" (Definition 1.2.11), "concept image" (Definition 1.2.17), and "A Hypothesis in the Conclusion" (Theorem 1.4.7). They turn out to be very useful. (There are many terms from logic we do not introduce. We only use terms that are actually useful for math.)

Chapter 1. Introduction to Proofs

Section 1.1 is a preview of proofs. The rest of Chapter 1 introduces terms from set-theory and useful logical equivalences.

Section 1.1. Preview of Proof

Why? This section covers a great deal that will be covered throughly in later sections [The section number is noted in brackets]. **With these terms and concepts you can discuss almost any part of math and logic whenever it comes up. The students will at least have heard of it and you won't need avoid math just because you have not lectured on the topic yet**.

For example, truth-table logic usually precedes the logic of quantifiers, but most mathematical conditionals are generalizations (e.g. "If $x > 2$, then $x^2 > 4$.") With the terms in Section 1.1, you can discuss generalizations even before the sections devoted to them [Sections 2.1 and 2.2]."

Think of Section 1.1 as motivating the study of logic for mathematics.

For example, mathematicians commonly think of sentences in alternative forms (e.g. the contrapositive is equivalent to the original conditional, Section 1.5). Theorem 17 (A Hypothesis in the Conclusion) is an alternative form which is very commonly used throughout mathematics (perhaps not with this name). This theorem is followed by an example which shows how theorems about "form" are used to reorganize theorems and their proofs. [Logical form is discussed in 1.3 through 1.6.]

Many exercises such as A1ff and B5-B42 can be done aloud in class. For example, B5-14 focus attention on how variables are used in mathematical sentences.

Section 1.2. Sets.

Why do Sets Now? Set theory terms are defined with logical connectives (e.g. *intersection* is defined with *and*), so **set theory and logic reinforce each other**. The logic in Sections 1.3 through 1.6 has perfect set-theory parallels and similar Venn diagrams. Sets of numbers (our universal set will almost always be the reals) make good examples for logic.

Think of this section as including set theory content, but also beginning lessons about connectives ("and", "or", "not", "if...then...", and "iff") and about how mathematical definitions are phrased.

Observations. Perhaps surprisingly, many students do not know much about sets and set notation when they enroll. They don't know the difference between [, (, and {. They don't know whether ∞ is a number. Students often don't know how to pronounce what they read, which makes reading harder. Experience shows that many students cannot read aloud "$\{x \mid x > 7\}$" and when they learn how, their comfort level increases. Interval notation is hard to read out loud. You might as well admit it. How do **you** pronounce "$(a, b]$"? After discussing pronunciation you may ask them exercises A9ff.

Long experience reading student's work shows that many write ungrammatical expressions and sentences (e.g. "$7 \subset (0, 10)$"). "Grammar" homework helps them keep track of the category of mathematical objects our symbols represent (1.2, A33ff).

Proofs are sequences of sentences. Therefore, we prefer to define terms in the context of sentences. Definitions of *intersection* and *union* are given in "**sentence-form**" (our term) which emphasizes the most useful way for proofs. (Section 2.3 shows that this is very useful.)

Proofs use "**concept definitions**" (Definition 1.2.17). Students develop "concept images," which are useful for understanding. But, to do proofs they need to use the actual definitions – in sentences.

<u>Our Universal Set</u>. Of course, sets can be very general and contain elements such as "Winston Churchill," but such sets rarely appear in mathematics. For almost all our work, **the universal set is conventionally the set of real numbers**. This permits us to use lower-case letters for numbers and upper-case letters for sets. Students are often initially confused about the difference between *membership* and *subset*. By keeping sets rotationally distinct from numbers, we can help clarify the difference.

Later, when we need sets of sets, this problem will have already been solved and we can begin using upper-case letters as elements. But it is not a good idea to do so when sets are first introduced.

<u>Conventions</u>. **We distinguish between the uses of lower-case and upper-case letters**. Variables for numbers usually are lower-case letters such as x, y, a, b, and c. (Some well-known formulas such as "$C = \pi d$" use upper-case letters for numbers.) The letters n, i, and j are reserved for integers. Sets are represented by S and T and upper-case letters near them in the alphabet. Sentences in logic are represented by upper-case letters such as H, C, A and B. Making the distinction firm helps students read mathematical sentences. (We avoid the "p" and "q" of logic texts; lower-case letters look too much like numbers.)

Lower case and upper cases letters commonly denote distinct things. For example, in integration in calculus, textbooks distinguish between f and F in $F(x) = \int_a^x f(t)dt$.

Do not ignore mistakes like using “A” for “*a*”. Cure this problem early. Our conventions are listed on page 44 of Section 1.3.

General Comments on Logic.

We emphasize only that part of logic which is useful **for mathematics**. Chapter 1 covers useful logical equivalences and deduction (using truth tables as a tool, not as an end). Chapter 2 continues with generalizations and existence statements. Students find the material in Chapter 1 on truth tables complete and satisfying. Most are able to construct truth tables easily. The focus is, however, not the ability to construct truth tables. The focus is the logic itself.

In logic we study form, not meaning. Form is abstract. Most students are much more comfortable with studying meaningful sentences with terms they understand than studying (meaningless) forms. Nevertheless, the emphasis is on the patterns, not on the examples of the patterns.

We name the important logical equivalences. They may not be names you learned before, but we assure you that you will grow to like our names. They make sense. By the way, there are many terms from logic we do not employ. If you have taught from a logic book, look at Appendix A at the end of this manual to see why we omit many terms from a philosophy course on logic.

Outlaw “It”. At first, students have trouble seeing the three (3) sentences in "$H \Rightarrow C$". They may say “It is true” and you may not be able to tell which “it” they have in mind. We get them to say “the conclusion is true” or “the hypothesis is true,”or “the conditional sentence is true,” whichever they really mean.

Too many students focus on the conclusion. For example, given that “If $x > 5$ then $x^2 > 25$" is true, many originally think that it says “$x^2 > 25$" is true. No. The compound sentence asserts that **if** $x > 5$, **then** $x^2 > 25$. Both the hypothesis and conclusion are **open** sentences, the conditional sentence does not say either is true.

“It” is a word that gets class time. Students soon learn not to say “it” in the context of conditional sentences when they realize how ambiguous it is.

Comments. Students who skip columns in truth tables often get them wrong. Insist they show all columns.

The really tricky connective is “if.., then....” To understand the truth-table definition, they respond best to the “broken promise analogy” (page 38), but, in that type of example, their understanding is really more of social responsibility than the connective itself. Example 15 gives a **mathematical** reason why "$\Rightarrow$" is defined the way it is.

The connective “or” is tricky in the context of solving equations. Is the solution to "$x^2 = 4$" given by "$x = 2$ and $x = -2$" or by "$x = 2$ or $x = -2$"? It is also not easy for some students to distinguish "*A* and *B* implies *C*" from "*A* or *B* implies *C*."

The connective "iff" is pretty easy for them.

The patterns from Chapter 1 are used repeatedly later. As an instructor, you want them to learn the theorems by name and to be able to use (and state) them correctly, but they will have much more practice with them in significant contexts later, so students who have trouble with patterns now will be able to learn them later.

By the end of Chapter 1 they are much more aware of the importance of *patterns*. But many do not yet think at the pattern level. Their thinking is not yet that abstract. But, they are much closer.

Section 1.3. Logic for Mathematics.

Section 1.3 begins truth-table logic. Of course, good examples from mathematics will be generalizations (the subject of Chapter 2), so we mention generalizations anyway in anticipation of Chapter 2. Here are some good examples.

<u>Good Examples of Logical Patterns</u>. Here is a list of examples (**in bold**) you can use again and again. They can be manipulated and expanded using the patterns in the logical equivalences.

Example 1A: $\boldsymbol{x > 5 \Rightarrow x^2 > 25}$ — true $H \Rightarrow C$

1B: $x^2 > 25 \Rightarrow x > 5$ — false, because of a counterexample, the converse, $C \Rightarrow H$

1C: $x \leq 5 \Rightarrow x^2 \leq 25$ — false not $H \Rightarrow$ not C

1D: $x^2 > 25$ and $x > 0 \Rightarrow x > 5$.

Example 2: Either half of $\boldsymbol{|x| < c}$ **iff** $\boldsymbol{-c < x < c}$

A theorem on absolute values ["$-c < x < c$" abbreviates "$-c < x$ and $x < c$," and it is good for them to become aware of abbreviations.]

Half of that is [split "iff" theorems in two by Theorem 1.4.5]

$-c < x < c \Rightarrow |x| < c$ [A and $B \Rightarrow C$, A alone is not enough]

Negate both parts to get

$|x| \geq c$ iff $x \leq -c$ or $x \geq c$. Another theorem on absolute values.

$|x| \geq c \Rightarrow x \leq -c$ or $x \geq c$. [half of the previous result]

$|x| \geq 5$ and $x > -5 \Rightarrow x > 5$. [from "Or" in the Conclusion]

(H and A) $\Rightarrow C$ does not imply $H \Rightarrow C$

The previous result does not imply "$|x| \geq 5 \Rightarrow x > 5$"

Example 3: $\boldsymbol{ac = bc \Rightarrow a = b}$ **is false**

Which can be fixed:

$ac = bc \Rightarrow a = b$ or $c = 0$ true

$ac = bc$ and $c \neq 0 \Rightarrow a = b$ ["Or" in the Conclusion, T 1.5.6]

A or $B \Rightarrow C$ is enough to show $A \Rightarrow C$, by Cases

"$a = 0$ or $b = 0 \Rightarrow ab = 0$" does assert "$a = 0 \Rightarrow ab = 0$"

Continue to use the ☺ examples aloud in class, e.g. A1ff, A13ff, A29ff.

Notation. There are many notations for "not." We spell out "not." You may find it convenient to introduce an alternative short hand (perhaps ~ or ¬) for board work and homework. For "and" we are equally comfortable with the whole word, "and," and the symbol, "∧", which is really only found in logic, but looks like a capital "A" and is therefore easy to learn. Later, when it comes time to state mathematical theorems, the word "and" is obviously preferable.

In class questions. There are many good in-class questions. Open the text to any homework section and begin asking problems marked with ☺.

Double negatives (re T1.3.16): Children should not be left unattended.
[A movie recommendation] Don't miss it.
(Triple!) Don't fail to miss it!

Section 1.4. Important Logical Equivalences.

The logical equivalences emphasized here are genuinely important because mathematical sentences are often reorganized using these alternative logical forms. They provide "sentence synonyms" that every mathematician must know.

Theorem 1.4.7, "A Hypothesis in the Conclusion" is extremely important for higher math. It is very useful in proofs (e.g. Example 10, Section 1.1, pages 7-8). Many proofs use the rearrangement of that logical equivalence. They often don't see the point of giving "*B*" first, but there is a good reason for doing so and they will grasp it later. We ask them to memorize it with "*B*" before "*H*."

Theorem 1.4.8, "Cases" mixes "and" and "or" in a way they do not find natural. "Cases" helps them understand the meanings of "or" and "and".

Observation: Experience shows that students who skip columns (combining two or more connectives into one column) often get it wrong. As I write this I just graded a homework with an 8-row truth table. Everyone (5 of 28) who skipped columns got the truth table wrong. All of the others got it right. Lesson: Make a column for every connective!

Section 1.5. Negations.

Negation of a conditional sentence needs a lot of emphasis.

Of all the logical equivalences, Theorem 1.5.2, **"Negation of a conditional sentence" is the hardest**. It is best to anticipate existence statements and go straight to the negation of **"For all x, $H(x) \Rightarrow C(x)$"** (page 62). They have an intuitive feeling for a "counterexample," but often cannot see the *pattern* in the concept. That is, they can say why "$x^2 > 25 \Rightarrow x > 5$" is false, but they don't see it abstractly. It is false because **there is** a case where the hypothesis is true **and** the conclusion is false. This is the most dramatic instance of students being able to **do** the right thing in particular examples without being able to give an abstract formulation of what they did.

The negation of "$H \Rightarrow C$" is "H **and** not C" (Theorem 1.5.2, page 62).

"The hypothesis is true **and** the conclusion is not."

You will often see them write that the negation of "$H \Rightarrow C$" is "$H \Rightarrow$ not C", or some other incorrect permutation. **We recommend you anticipate existence statements** as in Section 2.2, Theorem 6 (page 62 and page 105). In the context of a generalization, "For all x, $H(x) \Rightarrow C(x)$" is false precisely when "**There exists** an x such that $H(x)$ is true **and** $C(x)$ is false" (Section 2.1). Then you can explain why "$x^2 > 25 \Rightarrow x > 5$" is false. You need the "there exists" for counterexamples to make sense. This anticipation does not bother students. The *and* is very hard to remember. Many students wrongly insert "=>" instead.

"$x^2 > 25 \Rightarrow x > 5$" is false (as a generalization). That does not make "$x^2 > 25 \Rightarrow x \leq 5$" true.

It does make "**There exists** an x such that $x^2 > 25$ **and** $x \leq 5$" true.

We state Theorem 1.5.8, "Proof by Contradiction"but we do not emphasize it at this stage. Section 3.4 covers it thoroughly.

Section 1.6. Tautologies and Proofs.

Most students have an easy time with tautologies.

The preview with actual proofs and manipulation of forms is harder. It needs to eventually be mastered, but maybe not yet. It is good for the students to know where they are going. The exposure is important, but they need Chapter 2 before we can hope for mastery. However, this preview prepares them to expect to translate sentences (to exhibit the connectives) and then rearrange sentences (into useful logically equivalent forms).

They respond well to the idea that we do not need to distinguish between singular and plural (pages 75 and 104).

<u>Amazing Logic Difficulties</u>. Experience shows that some math students with good grades in previous courses nevertheless have trouble with simple deductions. As an instructor, **you cannot assume what is obvious to you is obvious to them**.

Conjecture: $x < 3$ implies $x \leq 3$.
Conjecture: $x \leq 3$ implies $x < 3$.
Conjecture: $3 < x < 5$ implies $x < 5$.
Conjecture: $x < 5$ implies ($x < 5$ or $x \geq 7$).
Conjecture: ($x < 5$ or $x \geq 7$) implies $x < 5$.

Most, but not all, students get these right as stand-alone problems, but more get them incorrect when the deduction is buried in a longer problem. Examples 4, 5, 8, 9, and 10 address what can be deduced from what. Exercises B23-31 (and Example 1 below) are good examples to do in class. We do not drop this topic. There are more such problems, of increasing complexity, in Sections 2.2 and 2.4, and you can easily make up more yourself.

Examples like this next one are wonderful for clearing up misconceptions (and there will be a lot of misconceptions!)

Example 1: Assume this is true: If $x < 4$, then $f(x) \geq 9$.

Which of these follow logically?

a) If $x < 3$, then $f(x) > 8$.

Some say "No" because $f(x) > 8$ does not imply $f(x) \geq 9$. "What about 8½?" But they are reversing the order. This needs to be straightened out. Lots of similar examples help (B27ff).

b) If $x = 2$, then $f(x) \neq 7$

c) If $f(x) < 10$, then $x < 5$

d) If $f(x) < 8$, then $x > 5$.

e) If $f(x) < 7$, then $x > 3$.

You can make more examples like this easily. Make sure your class has seen many and can do them. In problems like those you may permute the numbers and inequality symbols any way you like and some students will get them wrong until thoroughly trained. They are not as simple as good mathematicians think!

Chapter 2 has additional similar examples (e.g. 2.2, HW A31ff and B57ff). You can make up this type of example on the fly.

Example 2: Determine which conjectures **follow logically** (FL) from Assertion 1, or not (N). Assertion 1: If $x > 5$, then $f(x) \leq 3$.

a) FL N If $x = 4$, then $f(x) > 2$.

b) FL N If $x^2 > 100$, then $f(x) < 4$.

c) FL N If $x = 7$, then $f(x) \neq 4$.

d) FL N If $f(x) > 9$, then $x < 6$.

e) FL N If $x \leq 5$, then $f(x) > 3$.

Example 3: Suppose this is a fact: "If $x > 1$ and $t < 5$, then $f(x, t) \geq 3$."
If the following is also a fact, what can be deduced from those two facts?

a) $x = 2$.

b) $x > 3$ and $t = 4$.

c) $x > 2$ and $f(x, t) \geq 3$.

d) $f(x, 4) = 2$.

e) $f(x, t) < 3$.

We will have many more examples like these in upcoming homework exercises which you can do in class.

Examples 8 and 9 are very helpful. Do not be afraid to create examples that "obviously" follow, or do not follow, logically.

"$x < 4 \Rightarrow x \leq 4$." We have had math majors mark this "False." They ask, "What about the = ? It can't be = 4."

$|x| < 5 \Rightarrow x < 5$. "What about -10? x can't be -10."
In the **hypothesis**, x can't be -10, but that is not relevant.

You might be surprised to find out how many students get them wrong until they are trained.

Chapter 2. Sentences with Variables. Chapter 2 deals with sentences with variables. What do they say? How does logic apply to them? How can they be translated into other mathematical sentences which appear different, but express the same meaning? Mathematicians "translate" all the time, but hardly notice the process since they are fluent in Mathematics.

Students have seen some of Chapter 2 already in Chapter 1 because they have seen variables and connectives in action. Chapter 2 asks students to recognize and master all the mathematical usages of variables. This mastery is a tremendous accomplishment. They have learned to read!

Section 2.1 is on the three basic uses of variables. Sentences that look similar may have different interpretations. Section 2.2 is about existence statements and nested quantifiers.

Section 2.3 has a lot in it. It discusses placeholders and letter-switching. The Quadratic Formula is a great example.

How do definitions work? There is a very simple idea behind definitions, but that idea is truly profound: *equivalence*, which permits replacement. We want them to take a sentence with a new term they don't understand and translate it into (replace it with) an equivalent sentence expressed in more primitive terms. Then they can deal with it in those more primitive terms. However simple this is in principle, it is not their natural approach and they need to be reminded that it is the thing to do.

Students must learn to replace a vague understanding of a term ("concept image") with a formal "concept definition"(Definition 2.3.5, page 125). We use these useful terms to enable us to describe to students the level of understanding required to read and write proofs.

Section 2.4 points out how English can be ambiguous when Mathematics is not. They understand this.

Section 2.5 is on existence statements and form. The value of alternative forms from Chapter 1 is illustrated in the context of rational numbers. Conjectures are rearranged using our logical equivalences and existence proofs are introduced.

Section 2.1. Sentences with One Variable.

It is now time for students to realize what mathematical sentences are about. We hope they grasp the higher-level concepts of operations and order, and realize that some sentences are about such high-level mental objects.

We tell them (admit to them) that a lot of mathematics is written in a manner which does not explicitly distinguish the difference between two quite different kinds of sentences. They can understand that by analogy with the following example which we put on the board: What is the meaning of the word "lead"? If they answer about leading and following, we say it refers to the metal, and if they answer about the metal, then we mention leading and following. With a smile we then note that you need context to really know. Similarly with math. **Often you need context to know what symbols say, and the context may be your own prior knowledge**. They can deal with that.

They can better understand that it is their responsibility to decide what they are reading from the context when they realize that they do it all the time in English.

The role of generalizations in "true-false" questions is important, since it is fundamental to mathematical thought.

The term "open" is more useful than you might expect. In a conditional sentence, students often regard the conclusion as true, even though it is the entire conditional which is true. In the example, "If $x > 5$, then $x^2 > 25$," both the hypothesis and conclusion are open and not necessarily true. The term "open" gives us a way to express that to students.

<u>Student Errors</u>. Students may have difficulty distinguishing between quantified and free variables. For example, which letters can be changed in "If $x \in S$, then $x \in T$" without changing the meaning? Only "x". That sentence has the "for all x" suppressed. But "$b > 0$ and $c < d \Rightarrow bc < bd$" could have any or all the letters changed, since all are quantified.

Is it any wonder that mathematics is confusing to uninitiated students?

Parameters can be nicely introduced with some sequence of thoughts like this: Solve $2x + 1 = 7$. Solve $2x + 1 = 21$. Solve $2x + 1 =$ something. Well, we are always doing the same steps, so we could make it "$2x + 1 = c$" and solve. Now "c" is a parameter describing a family of related equations. Now move to "Solve $3x + 1 = 19$," and as many more similar problems as it takes. They should rapidly see the point. We could "Solve $ax + b = c$" and have a family of equations with 3 parameters and derive the solution of all similar equations once and for all. "x" and "c" play different roles.

Someone may ask, "Are parameters "free" variables?" It depends! We must have them in **sentences** (not just expressions) to tell. In "$x - a = b$" the parameters a and b are free. In "$x - a = b$ iff $x = b + a$" they are not free. The sentence "$x - a = b$" describes a number, x. The second does not. It describes a method--*how to solve* a type of equation.

This section pulls everything we have been talking about so far. Some terms are new (e.g. "existence statement”), but the thrust is consolidation and clarification of familiar ideas. Students are now sophisticated enough to be able to see why two sentences may look similar and yet be interpreted differently.

Something Worth Omitting. The difference between an identity (as an open sentence which is always true) and its corresponding generalization is subtle. Is

“$2(x + 5) = 2x + 10$” quantified or not?

It is not **explicitly** quantified. We regard it as **implicitly** quantified. Technically, it could be regarded as open. We do not belabor this in class. The question is, is it being regarded as one, or as many? Abstraction can turn the essence of many similar things into one thing. If "$2(x + 5) = 2x + 10$" is regarded as a separate sentence for each x, it is *many* (therefore, an open sentence which is always true). If it is regarded as an abbreviated generalization, it is *one* (therefore, a true generalization).

Section 2.2. Existence Statements and Negation.

The negation of a generalization is an existence statement. Some students will want to express the negation of “All are ...” with “All are not ....” This error must be addressed.

Example: Ask them to consider this statement: “All the people in this class are males.” That is false, but not because **all** are females, rather because **some** are females. Mathematicians use the word “some” but the use of “there exists” is more precise and it is the way the theory is phrased.

"$x^2 = y^2 \Rightarrow x = y$" is false, but so is "$x^2 = y^2 \Rightarrow x \neq y$." This illustrates that the negation of a generalization is not a generalization.

A Useful Term. Our term “positive form” (Definition 2.2.4, page 106) helps prepare the students for an important idea. We prove something false by proving its negation true. To do so we usually need the “positive form” of the statement in question. For the negation of some sentence, the simple negation, "not(some sentence)," is rarely useful. We want it in "positive form" where "for all" changes to "there exists," and vice versa, and the "not" is either gone or pushed as far inside as possible.

The comments about the (lack of) distinction between singular and plural are usually well-received by the students (pages 75, 104).

Make sure they learn to disregard the normal English inferences from singular and plural. In math, singular may refer to more than one, and plural may refer to only one.

Class Question: Conjecture: There exist element<u>s</u> of {1,2,3} in {2,4,6,8,10}.

Answer: Yes, even though there is only one, since we disregard plurals.

Student Difficulties. Generalizations can have implicitly quantified variables, so some students fail to note that the negations have a "there exists". For example, the problem: State the negation of "$x^2 > 25 \Rightarrow x > 5$," might yield the answer, "$x^2 > 25$ and $x \leq 5$," with the quantifier incorrectly omitted.

A hard problem could be: Give the negation of "For all $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x-c| < \delta$ implies $|f(x) - L| < \varepsilon$." Even good students commonly forget to quantify the "x" in the negation, since it is not explicit in the original sentence.

Warning. We often write generalizations without an explicit "for all." That's okay. But students who omit "there exists" (or some synonym) are omitting something important. We recommend you consider the omission of "there exists" as an important error. On page 108, we entreat them to "Never omit *there exists*" if they mean "there exists."

Syntax and Nested Quantifiers. Syntax is that part of grammar that concerns how word order affects meaning. Nested quantifiers are a case in point. As an example you could use some set S, say S = [0, 1), and consider all permutations of sentences like these:

Class example: Let $S = [0, 1)$.
Consider all combinations of "there exists" and "for all" and $>$, $<$, $\geq$ and $\leq$.
e.g. For all $x \in S$, there exists $y \in S$ such that $y > x$. [True.]
The next sentence uses the same words but changing the order changes the truth value.

There exists $y \in S$ such that for all $x \in S$, $y > x$. [False.]
For all $x \in S$, there exists $y \in S$ such that $y < x$. [False.]
There exists $y \in S$ such that for all $x \in S$, $y \leq x$. [True.]
[Four more.]

Toward Calculus: Calculus proofs have many existence statements. Here is a good statement to begin with because it is simpler than most:

Let $f(x) = x^2$. There exists h such that if $3 < x < h$, then $f(x) < 10$.

This is a nice example (there are many) of first proposing a candidate and then showing it works. One candidate that would work is 3.1. 3.5 would not work.

A Teaching Idea: **"Tell me what to write."** We are beginning to do proofs. If you ask students about steps in class, often the answers are vague–so vague you cannot be sure the student is thinking about it correctly. Rather than accept a vague answer that has seeds of a good idea and finish the good idea correctly ourselves, we sometimes say "What would you write?" or "Tell me what to write," when we are transcribing it to the board. This forces the students learn to express themselves clearly.

For example, in 2.2 B41-4 there are two statements which we could prove or disprove. Even it the student merely answers "true" or "false" we can ask "Why?" (which is approaching a proof). Then, if we decide to have them prove their answer, in one case they will prove it false with a negation. "What is the negation?" If the answer is too vague, you can say, "What would you write?"

Section 2.3. Reading Theorems and Definitions. It is time for students to master placeholders. Theorem 1 is a good lesson, addressed further in problems B3-B6.

Placeholders are a critical concept for reading theorems. Example 2, on the Quadratic Formula, is an excellent lesson. Here are problems with increasing degrees of difficulty.

1) $3x^2 + 5x - 10 = 0$.
2) $-6x - 4x^2 = 30$ [a, b, and c are not left to right.]
3) $2x^2 + kx - 8 = 0$. [A different letter.]
   [They are easy up through number 3.]
4) $dx^2 + px = k$. [All different letters]
5) $bx^2 + 2cx + d = 0$. [The same letters in different places.]
6) Solve for p: $2p^2 + dp = 20$. [Solving for something other than "x"]
7) Solve for y: $2xy + 3y^2 + 2x = 50$. [Solving for something other than "x", even though there is an "x" in it.]

Point out how **place** (in an equation expressing operations) is the key idea, not the particular letter holding the place.

The key idea to reading definitions is that **Logic applies to sentences, not words**. Therefore, we prefer to define sentences containing vocabulary words, not just the words themselves. A word needs a context, which a sentence provides. Thus, when it comes time to apply logic to mathematical sentences containing vocabulary words (as in proofs), we do not replace just the vocabulary **words** with their definitions, rather we replace **entire sentences** with their definitions. We invented the term "**sentence-form definition**" (Definition 2.3.4, page 124) for definitions that define entire sentences by sentences expressed in more primitive terms. Its use is illustrated in Figure 4, page 127. Many proofs use the idea of Figure 4.

Example 19 is a very illuminating example. The ability to replace something that is not understood, such as n^*, with something equivalent which is understood, is a major part of fluency in Mathematics. If you put up, in a pseudo-quiz atmosphere,

"Definition: $n^* = n/2$ if n is even and $n^* = 3n$ if n is odd. Solve for n in $n^* =$

(10*)(5*),"
some students will read the odd-looking equation to be solved and stop dead. When we wander by and offer aid, they may well ask, "What's n*?"

Of course, that is the point of defining n*. It tells you 10* and 5* and, even, n*. If they would have the courage to **replace** 10* with 10/2 = 5 (because 10 is even), they would be part way through. Some students do not see "=" as permitting replacement. In many cases, it is just a symbol in the middle of a problem to do. The definition of n* with its "=" sign does not seem to trigger the thought that we can replace the unfamiliar n*, 10*, and 5* with more familiar expressions.

Many students look at the question "Solve for n in n* = (10*)(5*)," find 10* and 5*, multiply them, and stop. Finding a number, any number, is so satisfying, that they don't replace the whole equation to get "n* = 75." So, they don't continue on to solve for n (150 or 25). Again, we wish they would **replace** things properly, not just do operations.

Another example: Let x# = 2x if $x < 4$ and x# = $(x + 1)/3$ if $x \geq 4$.

Solve for x: x# = (1.5#)(5#)

There are two answers.

Here is another:

Definition: $x@y = 2x - y$. Solve for x: $20@x = (2@1)(3@2)$.

This takes a real understanding of placeholders because the "x" is in a different place.

Student Difficulties. Some students have trouble seeing the three (3) sentences in "A iff B". When they say "it" in class you need to discover which "it" they have in mind. Sometimes "it" is "A", sometimes "B", and sometimes "A iff B".

For example:

$S \subset T$ iff if $x \in S$, then $x \in T$.
the sentence being defined — its definition
|----------------the definition in sentence-form -----------|

It is possible to use notation to give definitions that do not appear to define sentences. For example, consider Definition 1.2.9B (page 16) of set intersection: "$S \cap T = \{x | x \in S \text{ and } x \in T\}$." That definition is correct and fine, but, a set is determined by its members. So that is only abbreviated way of telling us when "$x \in S \cap T$" is true. You can't prove anything about the intersection of sets without first exhibiting a sentence with the connectives displayed. If you think you can, it is because you are so fluent you don't even notice the translation. Students need to make the translation step explicit.

Students are warned (Example 2.3.12, page 125) that many authors (even of high level texts) use "if" in definitions when the really mean "iff." For example, "An integer is even if it is divisible by 2." For another, "b is a bound of S if, for all $x \in S$, $|x| \leq 12$." Both of these are technically incomplete for lacking the "only if".

What not to Memorize. "Interior point" is a term in Definition 2.3.21. It merely serves as a moderately complex term to use as an *example* (which could be skipped) of how a definition works. (Some students erroneously think they must memorize it because it is in the text.) They are to learn *how* definitions state what they state, not just *what* they state. This section is about the requirements of definitions in general, not about particular definitions.

Aside. A proof that "2 is not an interior point of [2, 3]" uses a remarkable amount of logic we have studied. You can easily skip the "interior point" example entirely, but if you do it, consider discussing all the logic in this negation. First you need to convert the English version to the version with letters and quantifiers. Then you need to negate an existence statement "There exist a and b such that $2 \in (a, b) \subset [2, 3]$" to get "For all a and b, $2 \notin (a, b)$ or $(a, b) \not\subset [2, 3]$" (students often do not see the "and" in "$2 \in (a, b) \subset [2, 3]$"), which by the Theorem on "or" is replaced with
"For all a and b, if $2 \in (a, b)$ then $(a, b) \not\subset [2, 3]$" or with
"For all a and b, if $(a, b) \subset [2, 3]$, then $2 \notin (a, b)$."
The proof will likely address one of these logical alternatives.

Advanced Question about existence: Ask the class to create a definition of "local maximum" giving them hints from pictures. Let them find the existence part (most won't).
$f(x)$ has a local maximum at c iff there exists $\delta > 0$ such that if $x \in (c - \delta, c + \delta)$, then $f(x) \leq f(c)$.

Students must have the concept that two sentences can be equivalent ("sentence synonyms"), that is, say the same thing without appearing exactly the same. So, for 2.3, students must:
1) be willing to replace one sentence with another, equivalent, sentence
2) see unfamiliar terms in their entire context, that is, in a sentence.
[We prefer not to define just words, rather sentences with the words]
3) If they are uncomfortable with a term, throw out the whole sentence with it and replace the sentence with another without the term.

Section 2.4. Equivalence. They can do this. This section is short.

Note how the plurals and the words *all* and *any* express generalizations, and the *has* and *is* may indicate existence statements.

The discussion in the subsection about "The Misuse of *Or*" is important. Later, Section 4.1, page 230, the "bad proof" for Conjecture 19B gives a simple example that deceives many students.

Another example: "$|x| > 5$ implies $x > 5$ or $x < -5$," which is not the same as "$|x| > 5$ implies $x > 5$" or "$|x| > 5$ implies $x < -5$." You cannot put the "for all" which is outside the first version on each of the two parts of the second. "or" and "for all" do not distribute like that.

Do not mention this logical equivalence in class, but if a students brings it up, you can be prepared. A truth table would show
A => (B or C) is LE (A => B) or (A => C). This is a misleading logical equivalence they should **not** learn because it is too easy to misuse with generalizations.

There is nothing wrong with using
"For all x, {(A(x) => B(x)) or (A(x) => C(x))}."
The problem comes when we put the "for all x" on each part. "For all x" does not distribute across the "or", as we showed with examples in the text and above.

Another (counter) example: $R \subset S \cup T$ does not imply $R \subset S$ or $R \subset T$. The "for all x" does not move past the "or". All the logical equivalences in the text do not have this problem.

Point out that "equivalent" is not the same as "logically equivalent". Yes, "logically equivalent" implies "equivalent," but sentences can be equivalent for other reasons besides logical equivalence. Sentences that are logically equivalent are equivalent solely because of the arrangement of the components and connectives, not because of meaning.

Section 2.5. Rational Numbers and Form.

"x is rational" is an implicit existence statement. This section introduces existence proofs. Many of the sentences are complex enough that they can be rearranged using our logical equivalences, so there is a lot of focus on the form of conjectures.

Chapter 3. Proofs.

Possible examples for lessons are given below. First we comment on the homework and the pedagogy.

Homework. Many conjectures in the text are unresolved and some theorems are not proved. Their resolutions and proofs will be requested in "B" problems of the same number. For example, the resolution of Conjecture 9 will be in problem B9.

Lecture Preparation. From here on through the rest of the course **it is possible to conduct most classes with relatively little lecture preparation**. You can just have the students do the next few proofs and unresolved conjectures. We recommend at least discussing the terms and doing of few of the given proofs to give the students the flavor of what they are to do, but it is easy to actively involve the students in the work.

Pedagogy. **Why do We do Inequalities First?** Proofs have two components, logic and "prior results." For a result to be "prior" it must be known prior to the result we are attempting to prove. By doing proofs with inequalities we are able to clearly list **all** the prior results and ask students to cite prior results. We do not need to refer to things students may or may not know from other courses. Also, the subject of inequalities has just the right mix of easy and not-quite-so-easy proofs.

Furthermore, Section 3.2 on absolute values has inequality results that are important in calculus, and not all of them are obvious. When teaching about proof, if students already know the results, it can be hard to get them to focus on the justification. By using an subject area that is both important and somewhat complex, we can put the focus on justification and truth. Examples 3.2.2 through 9 demonstrate that truth is not always obvious. Some of the conjectures are false!

Examples precede theory. The many examples in Sections 3.1 and 3.2 explain a lot. Section 3.3 gives the theory. Section 3.4 discusses indirect proofs by contradiction or contrapositive, and Section 3.5 does mathematical induction.

Section 3.6, "Bad Proofs," reproduces common mistakes students make and asks them to identify what went wrong.

Section 3.1. Inequalities.

This section begins with a definition of *proof*. It discusses the requirements. It gives a few simple proofs.

Many texts that discuss how to do proofs fail to state what a "proof" is. (We find this amazing, but true.) We define *proof*. We want the students to be able to write a proof; it only seems fair to tell them what it is we am asking them to create.

The key to the concept of proof is its two parts:

form (logic) and *prior results*.

They have been studying *form* explicitly ever since Chapter 1. The new part is to get them to respect the role of prior results. That is, there may be steps that students would like to use that are true but not legal; steps must be *justified by prior results*. The "list" approach (pp. 160, 182) is helpful in this regard.

This is why inequalities are such a great place to begin– we can list **all** the prior results.

One difficulty with good math students is to get them to distinguish "true" and "prior." A sequence of **true** statements from which we can deduce the result we want is **not** a proof unless the steps are **prior** results.

We introduce the idea of "fixing" a conjecture that is false. The idea is to state something similar that is true, and then prove it. If you put up false conjectures, students will often notice that they are true if some additional conditions hold. Have them learn how to state results by fixing false conjectures. Example 3 is good.

This section leads smoothly into Section 3.2 on absolute values.

Lesson plan. This works best if you have lots of board space so you can watch and help students (in pairs or threes) at the board. Beginning in Chapter 3, we would want to have a classroom with lots of board space.

Rapidly lecture on proof, through Definition 3. Slow down and do Theorem 4, noting the references back to prior results. Do Corollary 5.

Let them try to prove the first part of Theorem 7.

Do 8A for them (They are not yet ready to play by the rules, so you can use this one to show them how to carefully reference prior results.)

Let them work through 8Bff (ask them to not look at the given proofs). Interrupt at any time.

Ask them to continue with Conjecture 9 and Theorem 10.

If you have time, consider have students evaluate the arguments for Example 5 and "another argument" for Theorem 16. (These are exercises B17 and B16.) We will discuss "bad proofs" latter in Section 3.6, but it is good to get students to judge arguments already. Many have trouble making decisions. "Is this step okay or not?"

<u>Proofs are not unique</u>. For example, here is an
Alternative proof to Theorem 7: $a < b$ iff $b - a > 0$ by D3
iff $(-a) - (-b) > 0$ by PR0
iff $-a > -b$ by D3.

Alternative proof to Theorem 8I "$a > 0 \Rightarrow 1/a > 0$.
$a > 0 \Rightarrow 1/a \neq 0 \Rightarrow$ [by 8F] $(1/a)^2 > 0 \Rightarrow a(1/a)^2 > 0$ [by A1C] $\Rightarrow 1/a > 0$ [PR0]

Note: Theorem 26 is important, but difficult for them. They have trouble stating the contrapositive (The location of "for all" is unusual). The proof is easy after that, but students may need to see it many times.

A similar result (perhaps for an exam) would be:

$$\text{If, for all } \varepsilon > 0,\ x > c - \varepsilon, \text{ then } x \geq c.$$

One can almost duplicate the proof of Theorem 26, or one can use Theorem 26 in a clever way using placeholders and letter-switching.

Proof: For all $\varepsilon > 0$, $x > c - \varepsilon$, so $c < x + \varepsilon$. By Theorem 26, $c \leq x$. [with this x being that c and this c being that x.]

Proof: Thou Shalt Not! (Proof in higher mathematics)

by Warren Esty

To prove a **result**

1) Do not Begin your proof with the result itself. (It is not prior.)

[However, you may note it is logically equivalent to some other formulation, and that you intend to prove that second formulation instead.]

The statement of the result you wish to prove probably has some letters [For example, S or f]. Your proof may use those given letters.

2) Do not Use other, new, letters unless

i) they are representative of all of their kind, by definition (as a representative case) (Often introduced with the word *Let*, for example, "Let $\varepsilon > 0$."), or

ii) known to exist by some prior result (in which case you write "There exists ... by Theorem x.x), or

iii) newly (and clearly) defined in terms of given things (for example, "Choose $\delta = \varepsilon/5$" after ε has been given).

New letters that appear in an argument they must be clearly defined or quantified, or else the argument might go wrong at that stage. Be careful with the existence and generalization of new letters.

3) Do not "Let" a letter have a property without justification that it really could have that property.

[When we, for example, "Let x be in S," the letter "x" has not been previously used and is not given in the problem. It is a name for an arbitrary member of S. That type of "Let" is legal. However, if f is given by $f(x) = x^2$, we cannot "Let $f(x)+2 = 1$." We don't know there exists such an x (and there is not). "Letting" a new letter have two properties at once is illegal unless you can cite a reason that there really exists such a thing.]

4) Do not Use the same letter for two different things in the same proof.

5) Do not Use true assertions that are not clearly prior.

(Non-trivial assertions must be from our list of prior results. Truth is not enough. You are not allowed to use assertions just because they are true. Of course, much-lower level results can be used without citing them. If you don't know if some result you want to use is really prior, label it a "claim".)

6) Do not Use a theorem to prove itself.

7) Do not Omit "there exists" when it is meant.

8) Do not Use "=" to connect things that are not really =.

9) Do not Fail to use "=" if you really mean =.

10) Do not Construct cases instead of using general representative cases (when a result is supposed to hold in general).

11) Do not Give counterexamples which fail to give values to **all** the letters.

Section 3.2. Absolute Values. Don't think absolute values are easy. They are not.

We use many conjectures to force the students to think critically. We lecture less and have the students make more decisions and give more justifications.

If you are a fan of group work, this is a good section to ask for, or at least allow, group work. We send students in groups to the board.

If you want to do calculus applications of absolute values and nested quantifiers, some are the next section (pages 184-189 and B55ff on page 195). Many are in Chapter 9, "Calculus" (which we never get to in one semester).

Lesson Plan: Ask the conjectures in Examples 2 through 9 aloud in class. Request specific counterexamples to the false ones. Do a bit of Theorem 2 through 4, making sure students understand Theorem 4. Them turn the class loose on 6 through the end, resolving them one after the other. Assign as homework a selection that depends upon how far they got, but include Theorem 7, the Triangle Inequality, even if it was done in class. Conjecture 14 is hard.

Comment on the value of doing scratch work on a different piece of paper. Too many student expect proofs should be like algorithmic computations where everything you write is correct, useful, and in the right order. However, in order to find a proof we might write far more than we use and we might have to select the good parts and reorganize what we wrote.

Absolute values make a great topic. You will see students using logic incorrectly and have a chance to fix it. For example, some begin with the result to be proved and proceed to deduce a final sentence like "$2xy \leq |2xy|$," which is true. However, deducing a true sentence does not prove that the initial sentence was true. You can't begin with the thing to be proved!

Assign B39, which is a deceptive bad proof. Many students do not find the real error, which is in Step 4. There is a lesson here about doing two steps at once. From Step 2 to Step 3 is possible, but Step 4 applies the triangle inequality in a deceptive manner. So, skipping steps can be okay or not. At this stage, we recommend they not skip steps, at least until their errors are extremely rare.

Conjecture: $|x + a| < b \Rightarrow x < b + a$. [False: $x = 10$, $a = -10$, and $b = 1$.]

Logic in Proofs. You might lecture on this or skip it. Skip it unless you have lots of time. It illustrates the form of proofs when the statement to be proved has a complicated form.

There is a lot of logic in proofs that good mathematicians hardly notice, but beginning students may need to have explained. Here is an example where the proof organization utilizes a lot of logic from Chapter 1.

Correcting Conjecture 16: "$|x| = c$ iff $x = c$ or $x = -c$" (which was false) leads to the fix (Conjecture 17 and Exercise B17, which could be proved early in the section):

Theorem: "$|x| = c$ iff $c \geq 0$ and ($x = c$ or $x = -c$)."

How would your students know the logical organization for a proof?
You can lecture on this form.

We ask them to "Look at the form":

(1) iff [(2) and ((3) or (4))]

We expect two sub-proofs by "iff" (Theorem 1.4.5).
To show the "=>" half of the "iff", show, by "Two Conclusions" (Theorem 1.4.6) two sub-proofs:

(1) => (2)
and (1) => ((3) or (4))

where the second is already done in the text, below Conjecture 16, using the definition of absolute values with the two cases $x \geq 0$ and $x < 0$.

To show the other half, <=, of the "iff": [(2) and ((3) or (4))] => (1)
Expect "Cases" because of the "or" (Theorem 1.4.8).
by the Distributive Law (Theorem 1.5.15)
[((2) and (3)) or ((2) and (4))] => (1)
by Cases, show
(2) and (3) => (1)
and
(2) and (4) => (1).

So, the proof of this one theorem has several parts, each of which does not address the whole.

This may be an "obvious" organization to some students, but you may have quite a few to whom this is not obvious.

Section 3.3. The Theory of Proofs.

This section discusses the difference between and argument and a proof.

Why can we assume the hypothesis is true?

The term “representative case” proof (Definition 3.3.4, page 184) is very useful.

Students learn to appreciate the difference between “prove” and “deduce” (page 186).

Some of the proofs have nested quantifiers as in calculus (Example 9 can be altered to make similar new examples).

Comment:

Question: To prove "(A => B) => (C => D)" can we assume A is true? No. We may assume C is true, but not A. A is not a hypothesis, rather "A => B" is. That conditional can be true even if A is false. So we cannot assume A is true.

The application of this thought is widespread. Many examples using terms like *increasing* or *one-to-one* have conditionals in both the hypothesis and conclusion.

e.g. Conjecture: "If $g \circ f$ is one-to-one, then g is one to one." [Starting in the wrong place is misleading. See Conjecture 5.1.9 for a great example of bad rasoning.]

Section 3.4. Proofs by Contradiction or Contrapositive.

Proofs by contradiction and proof by contrapositive are discussed and compared. You might note that many proofs said to be "by contradiction" are really "by contrapositive."

A Difficulty. When you teach and grade proofs, there is always the problem that some students don't pay enough attention to what steps are legal (They might write down anything that comes to mind, prior or not). Sections 3.1 and 3.2 train students to care about what is prior and what is not. However, in Section 3.4 and later sections, we need common sense.

We need to assume that simple results are on the list prior to complicated results, even if they have not be explicitly stated. For example, suppose you wish to prove:

Conjecture: If $x > 0$ and $y > 0$, then $\sqrt{(x+y)} < \sqrt{x} + \sqrt{y}$.

The proof might use: Theorem: If $x > 0$, then $\sqrt{x} > 0$.

A few students (not many) would wonder what they can cite to show that. It is not given in the text. We tell them that the theorem is well below the level of the conjecture and we will accept it without proof. Using our term, they may "**claim**" it (2.2 Definition 10, page 110). However, in the right context we might ask them to prove the theorem, in which case it would have to be proven from results even more basic than the theorem itself. It would be proven from the definition of "square root".

Prove Conjecture 2.3.17 (B45 in 2.3, and B24 in 3.4): If S is bounded, then S^c is not.

Suppose, for contradiction, S is bounded and S^c is bounded.

Then $S \cup S^c$ would be bounded by Theorem 2.3.16, but $S \cup S^c = R$ which is not bounded, so we get a contradiction and the supposition is false.

HW: B4 is good. With a little work, B5 follows from it, but when we assign both on the same HW, many students do not notice the connection.

B8 is good ($x > 0$ and $y > 0$ and $x^2 + y^2 = 1$ implies $x + y > 1$) and B23 follows from it. When we assign both on the same HW, many students don't notice the connection.

A Lesson about Properties and Stating Math. Here is a possible lesson about properties, designed to help students learn to state mathematics and conceptualize terms. Here is most of class one day, done after 3.4 but it could be inserted anywhere after 3.2. The text does not suggest or require you do this.

I stated this definition of "interior point" on the board: "p is an interior point of S iff there exist a and b such that $p \in (a, b) \subset S$ ". I drew a number line picture to explain it.

I asked them to get out a piece of paper and state conjectures about *interior point* that might be true or false. Emphasis on "state." I think most had no idea what to do. I don't think I waited a full 5 minutes, but then I wrote on the board: "membership, subset, intersection, union, complement" and asked if anyone had a sentence about membership (the first term in my list).

1) “If S has an interior point, then S has at least 3 members.” [The actual first contribution. Unexpected, but true. I did not determine its truth, just moved on to get more statements.]

I waited for more, but didn't get a courageous student, so I put up

2) “If p is an interior point of S, then p is in S.”

After a bit more waiting, I asked for its converse and got it.

3) [the converse]

Then I asked for inverses

4) “If p is not an interior point of S, then p is not in S.

5) [the inverse of (3)]

I asked whether they were true. [I ask several students and often get conflicting answers/guesses.]

Then, looking at my list, I requested a statement with *subset*:

6) [I waited a while (no one answered) before I wrote the first hypothesis of this and waited for someone to fill in the rest] “If p is an interior point of S and $S \subset T$, then p is an interior point of T.”

I asked for a statement with *intersection*

6) “If p is an interior point of S and p is an interior point of T, then p is an interior point of $S \cap T$.”

7) Its converse

8&9) Ditto for union.

Back to number (1), we easily revised that to an infinite number of points. [I mention statements with complement later.] We could have done its converse, but didn’t.

After we finished those, I left them up and I put up the definition of accumulation point: "p is an accumulation point of S iff for each $\varepsilon > 0$ there exists x in S such that $0 < |x - p| < \varepsilon$."

This is harder, so I explained two examples, 1 is an accumulation point of

(0, 1) and 2 is not an accumulation point of {1, 2, 3}.

Then we went through the existing (2) through (9) again with the new term replacing the old term, then (1), and its contrapositive, (10), and at the end one more 11) "If p is an accumulation point of S union T, then p is an accumulation point of S or p is an accumulation point of T." ["or" instead of "and"]
[There is a similar one for "interior point" with a different truth value.]

"Accumulation point" is not an easy term so students often guess wrong about the truth of the conjectures.

That did not take all hour, but most.

It was lots of fun and, I hope, illuminating about stating mathematical properties of concepts. Lesson: If a term refers to sets and numbers, its properties may have to do with how it combines with set-theory terms.

I didn't get, or state, in either context: Conjecture: "If p is an xx point of S, then p is not an xx point of S^c." [The answers differ for the two terms.]

There are many other possibilities that no one volunteered and I did not bring up: "Every point in (c, d) is an interior point of (c, d)." "The set of interior points of $[c, d]$ is (c, d)."

No one mentioned "p is an interior point of S iff there exists $\varepsilon > 0$ such that $(p-\varepsilon, p+\varepsilon) \subset S$," which is a result mentioned earlier in 2.3.

^^^^^^^^

Some harder problems:
Prove: "$n^2 + kn + c$ is divisible by 3 for all n" cannot be true, regardless of k and c.
If it were true for some n, then the next case would concern

$$(n+1)^2 + k(n+1) + c = n^2 + 2n + 1 + kn + k + c$$
$$= n^2 + kn + c + 2n + 1 + k.$$

The difference would be $2n+1 + k$. Is this divisible by 3?
If this difference is divisible by 3, then the next difference would be $2(n+1) + 1 + k$ which cannot also be divisible by 3, since it differs by 2 from the previous difference.

Prove: "$2n^2 + kn + c$ is divisible by 3 for all n" cannot be true, regardless of k and c.
The next terms would differ by $2(2n+1) + k = 4n+2+k$.
If this difference is divisible by 3, then the next difference would be $2(2(n+1)+1) + k$
$= 4n + 6 + k$, which differs by 4 from the previous difference, so it cannot also be divisible by 3.

Section 3.5. Mathematical Induction.

Induction proofs are very difficult for students who have not been trained to respect the difference between a conditional sentence, its hypotheses, and its conclusion. The induction theorem has a conditional sentence as a hypothesis ("For all n, $S(n) \Rightarrow S(n+1)$"). To untrained students, the hypothesis of that conditional looks a lot like what they are trying to prove ("For all n, $S(n)$"). This is confusing.

Those two sentences are quite different. Furthermore, we do not assume "For all n, $S(n) \Rightarrow S(n+1)$" – we prove it, in order to satisfy a hypothesis of the induction theorem. When the two hypotheses are shown to be satisfied, the desired conclusion is proved.

The text has more than enough examples. You can use homework problems.

Example which can be done two ways: $1+4+7+...(3n-2) = n(3n-1)/2$.
One way: The usual induction. Ask students for the general term before giving it.
Another way: After having done $1+2+3+...+n$, the above sequence is three times it minus a sequence of 2's.

Conjecture: $(1-a_1)(1-a_2)...(1-a_n) \geq 1- a_1 - a_2 -...- a_n$. [This is B32]
Example: $(.99)(.95) = .9405 > 1-.01-.05 = .94$. It holds in this case. Does it always hold?

There are recursive definitions (Examples 2 and 4, B7-11). Here is an easy recursion example: Let $a_1 = 2$ and $a_{n+1} = (2a_n + 5)/4$. Claim: $a_n < 3$ for all n. [Actually, it is < 2.5, but 3 is easy and true] and $\{a_n\}$ is increasing.

A very basic induction: Here is a given theorem, a prior result for this problem: If two sets S and T are open, then $S \cap T$ is open.
Prove by induction this following theorem: The intersection of any finite number of open sets is open.
[The difficulty is in picking a good notation and making the role of "n" clear. Good students can do it easily, but your weaker students may not be able to state it well.]

There are additional similar examples that use *union* and that use *closed*.

Error on the base case: B47 may be deceptive. It is the base case which is misrepresented.

<u>Comment</u>: The proof of Bernoulli's Inequality, Theorem 3, has some steps missing the reasons. Exercise A1 asks for the reasons. Many students do not know the reasons, or, at least, do not clearly recognize that the two "Why?" questions have non-trivial reasons. The first needs the "$x \geq -1$" hypothesis and the second uses "$nx^2 \geq 0$."

At this stage, many students do not check every "=" to make sure it is true. They just read or skim them. They do not check "$\geq$" to be sure it goes the right direction. They may be aware of what was "done" to produce the step (like multiplying an inequality by $1+x$), but not whether doing it was justified (it is okay only if $1+x \geq 0$, so the hypothesis "$x \geq -1$" is inserted in the theorem and used in the proof).

Possible in-class work: B3 is harder than it looks. B17, B31, B35.

Additional Induction problems: $\sum_{i=1}^{n} i(i+1) = n(n+1)(n+2)/3$.

Any $n \geq 2$ can be written in the form $n = 2a + 3b$, where $a \geq 0$ and $b \geq 0$.

$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$, for $n \geq 2$.

$\sum_{i=1}^{n} i2^{i} = 2^{n+1}(n-1) + 2$

Easy: $2n + 1 < 2^{n}$ for $n \geq 3$.
This might serve as a quiz/exam problem to see if they know how to format an induction proof.

A hard (tricky) one, because the formulation is not typical: $\sum_{1}^{n} \frac{1}{k(k+1)} < 1$.

By induction, we could prove the left side is $n/(n+1)$, which is less than 1. However, the given version does not have "n" on the right and is not immediately suitable for induction.

Section 3.6. Bad Proofs.

In class we have had good results by asking students to "Judge every line. Put a check on it if it is okay, and an X if it is not okay. Read it line by line." We may have them "Check it with your neighbor."

Many students hesitate to place any checks or X's. However, they must learn to decide whether sentences are true or false, and whether sentences follow logically or do not follow logically. These exercises help.

Students, perhaps in pairs, can do these in class: Conjecture 3 and Argument 4. Conjecture 6 and Argument 7. Conjecture 10 and Arguments 11-12. We ask for checks on every line that is okay and an X where it goes wrong. We ask them to analyze lines sequentially. Each line is either okay or not, one after another. Conjecture 17 and Argument 18. Conjecture 19 and Arguments 20-21.

Notes: In Argument 18, line 6 does not follow. We ask what **would** allow it to follow.

Related (false) conjectures are: $x^2 \le y^2 \Rightarrow x \le y$. $x \le c^2 \Rightarrow \sqrt{x} \le c$.
Conjecture 17 can be fixed with both a and $b \ge 0$. The left sides does not even make sense if one is negative and the other positive.
Argument 21 is almost okay, but line 4 does not follow from line 3. It would, if lines 2 and 3 had $|a|$ instead of just a, which would have been legal. However, as stated, the right side of line 3 might have been increased by attaching absolute values to a., making line 4 not follow.

Good HW, or in-class work: B26, 28, 31.

B41 is a bad proof. However, it may be hard to find the exact error. Going from Case 1 to Case 2 is misrepresented. If $\max(a, b) = 2 = n+1$, we cannot assume both a -1and b-1 are natural numbers that fit case n (= 1), because $a = 1$ and $b = 2$ would have a-1 = 0 which is not a natural number. A very subtle mistake!

Part 2: Practice. Chapters 4 through 9. The text offers a lot of flexibility in Chapters 4 through 9. At Montana State and Stonehill we go straight through and finish one semester with Section 5.2 or 5.3. If you want to skip some sections and cover some later material, say, Number Theory in Chapter 6, the first page of the chapter tells which earlier sections are necessary prerequisites.

We think it is time for the students to do much of the work. If you lecture, you could "cover" the material rapidly, but covering the material is not the same as the students learning it. If have the students do the work, each section can take longer, or much longer.

About Chapters 4 through 9.
Class Preparation and Homework.

In any given section, perhaps define a term and select a corresponding theorem to prove, as an example of a proof in that subject. You might talk them through a proof or two with the book open. Then ask some of the easiest ones aloud (and there are some that are very straightforward).

Then have them work through the theorems and conjectures, one after another. It is very easy, and remarkably effective, to conduct class by having students do much of the work, possibly individually but preferably in small groups at the board. You can help individuals, groups, or the whole class, as appropriate. Then, depending upon how far they get and how much help they need, the next homework assignment can be to resolve some remaining conjectures, or additional problems from the exercises.

In Chapter 4 and following chapters the exposition has sufficiently many theorems and conjectures to conduct class just by going through them, one after the other.

The End of One Semester. At Montana State University and at Stonehill College the 3-credit sophomore-level course covers through 5.2 or 5.3 (5.3 on Cardinality is interesting.) At MSU we have had a subsequent course that resumes with Number Theory.

The text offers a lot of flexibility in Chapters 5 through 9. You could skip Chapter 5. You could continue immediately with Number Theory (Chapter 6), Topology (Chapter 8), or Calculus (Chapter 9). Only Chapter 7, Group Theory, has a prerequisite from an earlier chapter. It requires some number theory about modular arithmetic from Chapter 6.

Chapter 4. Set Theory

Chapter 4.1 is excellent for work with alternative forms. After 4.1, Section 4.2 is useful in Real Analysis (a.k.a. Advanced Calculus), but almost any chapter could follow next.

<u>Lesson Plan for 4.1</u>: You can ask class to identify the results from logic which parallel the theorems in 4.1. "What is the logic result corresponding to Theorem 1A?" 2A? 3? etc.

Each of the early results in 4.1 is a perfect parallel to a tautology, contradiction, or a logical equivalence.

Prove Conjecture 11 from the definitions.

Reconsider Conjecture 11 and give advice about proofs. The organization of proofs about *subset* (and many other terms) often follows from this general advice

1) identify the conclusion (its concept definition)
2) translate (if appropriate), and
3) reorganize, using a logical equivalence, if appropriate.

However, if the term is new, such as *upper bound* or *bounded above* or *least upper bound* in Section 4.2, the concept image may need substantial work before the ideas for a proof can come. The above advice still holds. With practice, students will eventually automatically do those steps.

The (bad) "proof" for 4.1, Conjecture 19B is an excellent example of "the misuse of *or*" which was first discussed in Section 2.4, page 145. Be sure the students understand this.

Continue with students working on the following conjectures. Assign homework depending upon how far they got.

Additional problem: Conjecture: $S \subset T$ or $R \subset T$ implies $S \cup R \subset T$.

Section 4.2 has much more in it than 4.1. The concept of supremum is not trivial. Again, the students can do much of the work.

4.2. Homework B1 discusses a difference between Math and English. “Upper bound” has “upper” in a position that would make it an adjective in English, but it is not an adjective modifying a noun in Math. In Math, it is part of a two-word noun.

4.2. Argument 34A and Argument 34B are good exercises. Some accept line 5 of 34A when they should not. Most students accept line two of 34B when they should not. Here are some things "Thou Shalt Not" do in proofs (next page). You might let student assemble their own list of advice that works.

Possible Additional problems:

4.2. After Conjecture 30. Possible fix with an additional hypothesis: “If M is an upper bound of S and ...[continue as Conjecture 30] and then reread Argument 30A. Would it be a proof? [Yes.]

4.2, right after Conjecture 36: Conjecture 36B: If f(x) ≥ 0 and g(x) ≥ 0 for all x, then sup{f(x)g(x)} = sup{f(x)}sup{g(x)}.

4.2. Theorem 37 has “=”. It is easy to prove one way, $\leq$, but not so easy to prove the other.

4.2. T F sup($S \cup T$) $\leq$ sup(S) + sup(T).

4.2. T F $S \cup T \in \boldsymbol{P}(S) \cup \boldsymbol{P}(T)$. [$\boldsymbol{P}$ denotes the power set.]

Grammar: $\{\varnothing\} \subset S$. $\{\varnothing\} \subset \boldsymbol{P}(S)$. $\varnothing \in S$. $\varnothing \in \boldsymbol{P}(S)$

Chapter 5. One-to-One and Onto.

Section 5.1 has two wonderful terms, *one-to-one* and *onto*. They have not seen many terms defined with existence statements, so *onto* is hard at first.

Chapter 5 has two wonderful examples of bad proofs. Be sure to cover the argument for 5.1 Conjecture 9 (pages 258-259) and 5.2 Conjecture 6 (page 265). The point is, when a conjecture is supposed to hold “for all” cases, use a **general**, fully representative, case. Do not **construct** cases.

Note: In Section 5.1 on one-to-one and onto, Conjecture 9 in Section 5.1 is false. The argument for it is very deceptive and illustrative of an extremely important point: Start in the right place! Be sure your students understand the subtle error in the proof. This is strong motivation to begin with the hypothesis in the conclusion, which we emphasize. Beginning elsewhere often leads to errors.

The proof was supposed to prove something "for all b_1 and b_2," but it did not start there. It did not address **all**. Each of the steps would work if there existed a_1 and a_2 such that line 2 was possible (which could be fixed by assuming f is onto), but that is not known. A principle is: Do not introduce letters (such as a_1 and a_2 in Conjecture 5.1.9) without their being

1) already given in the problem, or
2) representative of all of their kind, by definition, as a representative case, or
3) known, and asserted, to exist by some prior result, or
4) newly (and clearly) defined in terms of given things

5.2. Conjecture 6 has a deceptive argument. Be sure to cover it and discuss how to avoid such mistakes.

5.2. Conjecture 20A is false but with a simple misleading picture it looks true. This can be a very good lesson in the value of sticking to concept definitions. Sketch a graph like $y = x^2$ and let W be a set of y-values above the x-axis. Then consider any element, y, of W. There will be an x such that $f(x) = y$. Then x is in $f^{-1}(W)$ and y is in $f(f^{-1}(W))$. In this example W will be a subset of $f(f^{-1}(W))$. But the "there exists" step is not justified in general without an extra hypothesis: If f is onto, then ... .

Extra, not easy: Let f:A → A and g: A → A (all sets are the same, A).
Resolve the Conjecture: If $g \circ f(x) = x$ for all x in A, then g is one-to-one.

It is false. Let A = [0, 1], $f(x) = x/2$ and $g(x) = 2x$, for $x \leq ½$ and $2x - 1$ for $x > ½$.

A student's bad proof of Conjecture 8: If $f(S) \subset f(T)$, then $S \subset T$.
line 1) Let $y \in f(S)$. 2) Then $y \in f(T)$. 3) $y = f(x)$. 4) Then $x \in S$ and 5) $x \in T$. 6) So $S \subset T$.

Another deceptive bad proof (of a false concjecture):
Definition: $y \in f(S)$ iff there exists $x \in S$ such that $y = f(x)$.

Conj: $f(S) \cap f(T) \subset f(S \cap T)$

"Proof": Let $y \in f(S) \cap f(T)$
then $y \in f(S)$ and $y \in f(T)$
so there exists $x \in S$ and $x \in T$ such that $f(x) = y$ [by definition of $f(\ )$]
so $x \in S \cap T$ and $f(x) = y$
so $y \in f(S \cap T)$.

Chapter 6. Number Theory. Number theory deferred it to Chapter 6 only because a few key results, like the Remainder Theorem (6.2.2) are a little long and we wanted our students to have more exposure to straightforward proofs first. Nevertheless, you could skip to Number Theory right after Chapter 3 if you prefer. If your time is limited, Section 6.1 through 6.3 would be enough to give students the flavor of number theory. If you want to cover group theory, you should cover modular arithmetic in 6.4 as well. Students love the cryptography section (6.5). It is very real-world and interesting. To do it begin with 6.1 and go straight through to 6.5.

Additional Problem: If 2^n-1 is prime, then n is prime.

Chapter 7. Group Theory. This is like the beginning of an undergraduate course in Abstract Algebra. This chapter has a lot of material, even if only four sections, and could take many weeks to cover, or you could fly straight through it faster.

We would just go straight through the exposition and not introduce additional terms, but there may be results and terms that interest you that are not covered in the exposition. The exercises are designed to introduce such terms and give students related problems to work. So, look in the exercises for more on each topic.

Chapter 8. Topology. This is an introduction to the sort of topology used in Advanced Calculus and Real Analysis. It does not do any more-general topologies. It begins topology the same way it is begun in many analysis texts. (The definitions of topology can be organized in several distinct ways. We pick one common way.)

Again, there may be results and terms that interest you that are not covered in the exposition. The exercises are designed to introduce such terms and give students related problems to work. So, look in the exercises for more on each topic.

Chapter 9. Calculus. This chapter takes a beeline through the main limit and derivative results of calculus. It closely resembles an "Advanced Calculus" or "Real Analysis" course.

Chapter 9 could directly follow Chapter 3 on proof, but some of the proofs are a bit tricky and many math majors will take advanced calculus and don't need to see this material twice. At Montana State University our Mathematics Education majors are unlikely to take full courses in Number Theory, Group Theory, or Topology, so we put those chapters first so that they can study them in a second-semester course from the same text.

Appendix A: Terms from Logic to Omit

Do not bother to read this unless you are familiar with logic from some other text. This is about names for results from logic.

Unfortunately, many results in logic have more than one name, and most of the names from a logic course are not illuminating. Furthermore, most of the usual names are not used (or even known) by regular mathematicians. Thus we have chosen to use illuminating names, keeping the students and mathematical practice in mind. For our own curiosity we have compiled a list of alternative names (found in philosophy courses). We recommend you omit them.

Note: In logic, a “horseshoe” ($\supset$) is a symbol for the mathematical "=>". This is very unfortunate for Mathematics, because the horseshoe points the wrong way for the correspondence between subset ($\subset$) and "if..., then...” (pages 23, 44). We haven't yet had any students comment on this, and we certainly do not comment on it (Why confuse them?) but those who have taken logic might find this reversal confusing at first.

The term “hypothesis” is similar in Mathematics and in logic, but not quite identical. In logic, a “hypothesis” is a conjecture offered as a possible explanation. In Mathematics, a “hypothesis” is the "A" part in a conditional sentence of form "A => B." If the conditional is true, the hypothesis may serve as an “explanation,” but the true conditional would not be have “A” as a “possible explanation,” rather a sufficient condition.

We avoid the “p” and “q” of logic texts; lower-case letters look too much like numbers

Chapter 1 is about the “Propositional calculus” and Chapter 2 about the “Predicate calculus.” Neither term is at all illuminating. (What does this have to do with the students' conception of "calculus"? Who knows what “predicate” means?)

text usage	alternative omitted usages
open sentence	propositional function, predicate
variable	indeterminant
"A and B"	the conjunction of A and B
"A or B"	the disjunction of A and B, alternation, inclusive disjunction
"A => B"	implication [this term is misleading, because in English the “implication” is the conclusion, not the whole sentence. We call "A => B" a "conditional sentence."]
"A iff B"	biconditional, equivalence [this is misleading without variables. How can 2+2 = 4 be equivalent to the Fundamental Theorem of Calculus? Both are true, so "2+2 = 4 iff FTC" is true, but no real “equivalence” is expressed.]
conditional sentence	implication
hypothesis	antecedent, premise
conclusion	consequent
contradiction	absurdity

transitivity of "=>"	law of syllogism (in logic, a “syllogism” is any statement with two premises and one conclusion)
Modus Ponens	law of detachment, law of the excluded middle
contrapositive	transposition
proof by contrapositive	Modus Tollens, law of contraposition
proof by contradiction	law of conjunctive inference
	law of conjunctive simplification
	reductio ad absurdum