Final Exam on *Proof*. Name:__________________
Sections 1.1-5.1. Spring 2013 initial each sheet of extra paper

1. (9 pts) Give the sentence-form definition:
a) power set

b) union

c) onto

2. (8 pts) These might have grammatical mistakes or unconventional usages. If not, just say it does not. However, if it does, indicate what is wrong.
a) $x \in S \leq \sup(S)$.
b) $\{x\} \in \mathcal{P}(S)$ [$\mathcal{P}(S)$ is the notation for the power set]

c) $x \in S$ and $x \in T \Rightarrow x \in S$
d) $S \cup T \Rightarrow x \in S$ or $x \in T$.

3. (6 pts) Do these two sentences have the same meaning? Yes (Y) or No (N).
a) Y N Let $f(x) = 2x$ for all x. Let $f(a) = 2a$ for all a.

b) Y N $2x + 3x = c$ $2x + 3x = d$.

c) Y N $S \subset T$ If $x \in S$, then $x \in T$

4. (6 pts) Is the sentence open or not?
a) If $x \in S$, then $x \in S \cup T$.
b) $S \subset S \cap T$
c) $x(x - 1) + x = x^2$.

5. (6 pts) State the logical equivalences we know by these names:
a) A Version of the Contrapositive

b) A Hypothesis in the Conclusion

6. (8 pts) Give the contrapositive of these.
a) If $c > 0$, then there exists x in S such that $x > \sup(S) - c$.

b) If $x > c$ for all $c < b$, then $x \geq b$.

7. (12 pts) Make a truth table, with a column for each component (A on the left) and each connective, to determine if "A or not B" is logically equivalent to "$B \Rightarrow A$." Then state if they are logically equivalent or not.

8. (16 pts) Give the negation (in positive form) of these.
a) [f is given] $f(x) < 8$ for all $x > 2$.

b) [f is given] For $b > 0$ there exists m such that $x > m$ implies $f(x) > b$.

c) [About piles of chips] At least one pile has at least 10 chips.

d) $ax^4 + bx^2 = k$ has a solution.

9. (4 pts) Prove this is false: If $x > 5$ or $y > 5$, then $xy > 25$.

10. (8 pts) Theorem: $1 + 2 + 3 + 4 + ... + n = n(n+1)/2$.
Use it to derive a formula for the sum of the even numbers from 2 through k, where k is even.

11. (9 pts) True or false? No reason required.
a) T F If $f(x) = x$ and $g(x) = x^2$, then $g \circ f$ is onto $\mathbb{R}$, the reals.

b) T F If $0 < |x| < |c|$, then $|1/x| > |1/c|$.

c) T F If $|x - c| < d$, then $x > c - d$.

12. (12 pts) Suppose this is given: If $x < 3$, then $f(x) > 9$.
Which of these follow logically (FL)?
a) FL not FL $f(2) > 8$

b) FL not FL If $f(x) = 9$, then $x \geq 3$.

c) FL not FL If $x = 4$, then $f(x) \leq 9$.

13. (16 pts) Suppose this is given: If $x < 5$ and $y \geq 10$, then $f(x, y) < 15$.
What can you deduce from that and the following additional fact?.
a) $x < 4$ and $f(x, y) > 20$.

b) $f(x, y) = 16$ and $x = 6$.

c) $x > 6$ and $f(x, y) > 18$.

d) $x < 6$ and $y > 11$.

^^^^ **Proofs and Resolutions**
Instructions:
- Do proofs from the **definitions**.
- **Do not cite very similar results** to "prove" things.
- **Cite reasons** for each step. [If a step is algebra, just do it. However, if it relies on definitions from this course, you must cite the reason for the step.]
- Counterexamples must be complete.

(12 pts each) [Use the separate paper provided] Do all three of 14-16.

14. Let $f(x) = 3x - 5$. Prove: For $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - 16| < \varepsilon$ if $|x - 7| < \delta$.

15. Prove by induction. Do not use similar results to prove this. Cite reasons for every step to prove you know why that step follows. Steps without reasons will get less than full credit.

Theorem: If $0 < x < 1$, then $(1 - x)^n \geq 1 - nx$, for $n \geq 1$.

16. [Assume the two functions have the same domain and finite suprema. Do not cite any similar results. Use only the definitions. Correct steps without clear reasons will get less than full credit.]

Conjecture: $\sup\{f(x) + g(x)\} \leq \sup\{f(x)\} + \sup\{g(x)\}$

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17. (4, 4, 10 pts) Read and use this definition to do the following problems:
Definition: Let $f : A \to B$ be a function and S be a subset of B. $f^{-1}(S) = \{x \mid f(x) \in S\}$.

[Be careful. This is NOT the inverse function applied to numbers. It is exactly what the definition says it is, neither more nor less. Read the definition for what it says.]

a) Example: Let A and B are the set of all real numbers and $f(x) = x^2$. Find $f^{-1}([1, 2])$.
b) Restate the definition of "$f^{-1}(S)$" in sentence-form.
c) Resolve this Conjecture: $f^{-1}(S \cap T) \subset f^{-1}(S) \cap f^{-1}(T)$. [This one is in general, not about the example in part (a)].

Do TWO (2) of the remaining three. (12 pts each)

18. Conjecture [with the usual notation]: If $g \circ f$ is one-to-one, then f is one-to-one.

OMIT ONE of 18-20.

19. Conjecture [with the usual notation]: If $g \circ f$ is onto C, then f is onto B.

20. Conjecture: If S is not a subset of T, then the power set of S is not a subset of the power set of T.