PME 16 (1995)

PROBLEM SOLVING USING ARITHMETIC AND ALGEBRAIC THINKING

Anne R. Teppo and Warren W. Esty
Montana State University

A set of word problems was administered to a group of undergraduate precalculus students to study their use of algebraic and arithmetic thinking as a way to reason effectively about the given problems. An analysis of the students' responses indicates that, while many of these students were able to identify the mathematical relationships involved in the problem situations, they were unable to think about these relationships in an algebraic way. These findings are related to a model of algebraic thinking developed by Sfard and Linchevski (1994).

Contrasting Word Problems

Many students progress through four years of high school mathematics without developing appropriate levels of algebraic thinking. An understanding of the nature of this thinking is an important part of algebra reform (Kieran, 1992).

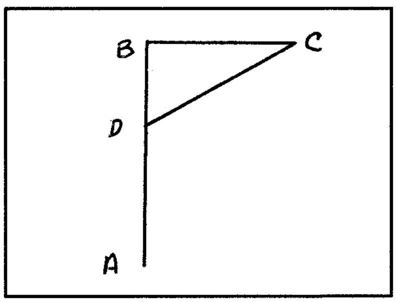
Two parallel sets of problems were developed that contained related forms of different word problems. One form of each problem could be solved using only arithmetic. The other form required algebraic thinking about the mathematical operations and the relationships between these operations. Such thinking could not be carried out unless some form of representation was used for the unknown quantities and the relationships given in the problem. Only after this representation had been accomplished could a student proceed to a numerical answer. In contrast, the arithmetic form of the problem could be solved by a series of sequential computations where operations were applied to known numbers using familiar formulas.

The questions used for each problem set are given below. In set I the algebraic problem involved understanding an income tax rate and the calculational problem dealt with a geometric figure. In set II the problem types were reversed.

Set I

1) In 1990 a family of four paid no taxes on its first \$13,650 of income. All income above \$13,650 was taxed at a 15% rate. If a family paid 7% of its income in taxes, what was its income?

2) (See Figure 1) Line segment AB is of length 2. Line segment BC is perpendicular to AB and has length 1. If AD is 1.3 units long, what is the sum of the lengths of AD and DC?



Set II

1) In 1990 a family of four paid no taxes in its first $13,650 of income. All income above $13,650 was taxed at a 15% rate. If the family's income was $18,500, what percent of its income was paid in taxes?

2) (See Figure 1) Line segment AB is of length 2. Line segment BC is perpendicular to AB and has length 1. If the sum of the lengths of AD and DC is 2.5, how long is AD?

The parallel problem sets were designed to act as controls for each other. Thus it was possible to test for students' understanding of the problem situation separately from their ability to use this information in an algebraic manner. When administered, the two problem sets were divided equally among 99 students enrolled in an undergraduate precalculus course. These students had previously taken three or four years of high school mathematics. The problems were given to the students two weeks before the end of the one-semester precalculus course.

Characterizations of Solution Strategies

The students' responses were analyzed by comparing and contrasting the solution strategies used for each problem. Indicators of solution strategy consisted of (i) the written expressions or formulas, in either numeric or algebraic form, that represented any of the mathematical relationships for the given problem and (ii) the uses made of these relationships to solve for the specified quantity. Student papers were separated into three categories; (1) correct solution or minor arithmetic errors, (2) evidence of some correct thinking, but containing one or two incorrect mathematical relationships, (3) no evidence of mathematical relationships appropriate to reaching a solution to the problem. Table 1 presents the distribution of student responses by category and problem type for both sets of problems.

Table 1 Solution Strategies as a Function of Problem Type

	geometric figure				income tax			
	1	2	3	total	1	2	3	total
arithmetic	47 94%	0	3 6%	50 100%	41 83.7%	7 14.3%	1 2%	49 100%
algebra	10 20.4%	20 40.8%	19 38.8%	49 100%	20 40%	16 32%	14 28%	50 100%

The distribution of responses in each category for the two problem sets shows a sharp contrast between the students' abilities to deal with mathematical relationships arithmetically and algebraically. For both the geometric figure and the income tax problem, the students (94 % and 83.7% respectively) demonstrated that they were able to understand the mathematical relationships in the given situation and correctly compute an answer as long as they could represent and deal with these relationships at an arithmetic (numerical) level. In contrast, when it was necessary to work with these same relationships at an algebraic (symbolic) level, only 20.4% and 40% of the students, respectively, were able to arrive at a correct solution. When students were denied a numerical access to the mathematical relationships, it appeared that they lacked the alternative algebraic way to think about quantity, operations, and order. It is postulated that this inability to think algebraically prevented students from understanding problems that were otherwise clear to them within an arithmetic framework.

In the geometric problem 47 of the 50 students answering the arithmetic question were able to select the appropriate relationships and evaluate these correctly. Even though these students were currently studying trigonometric relationships in the course, most of them (36 of the 47) used a more straightforward approach involving the Pythagorean Theorem with triangle BDC. When the same problem required an algebraic solution, however, only 10 of 49 students (20.4%) applied the Pythagorean Theorem to this triangle to create and solve an appropriate equation. None of the students who used trig managed to solve the problem.

Most of the unsuccessful students attempted to reach a solution using trigonometric relationships on triangle ABC. Because they could not perform direct numerical calculations on the two unknown sides of triangle BDC, they drew in and worked with triangle ABC for which trigonometric relationships do yield numerical, but irrelevant, calculations. These responses illustrate the inordinate lengths to which the students were willing to go in order to be able to find numerical computations they could perform.

The income tax question is interesting because it demonstrates students' abilities to handle a multi-step problem in which they must understand and relate two different mathematical relationships. In the arithmetic form 83.7% of the students were able to arrive at the correct answer, compared to 40% in the algebraic form. At the other extreme, only one student could not demonstrate at least a minimal understanding of the mathematical relationships involved in the arithmetic form of the problem, while 14 students were unable to make any headway when the problem was presented in an algebraic form.

The responses to the two forms of the income tax problem demonstrate the degree to which the students were competent problem solvers as long as they were able to operate arithmetically. At this level of thinking 83.7% were able to understand the problem situation, identify the mathematical relationships involved, select appropriate operations, and carry out the calculations in the correct order. However, when these same mathematical operations and order had to be represented and manipulated symbolically rather than used computationally, 60% of the students answering the algebraic from were unable to successfully solve the problem. It is postulated that these students lacked some form of mental representation for the missing numerical values that would have allowed them to reason about the problem situation. Similar behavior has been reported by Simon and Stimpson (1988) where students gained access to reasoning about unknown quantities through diagrammatic representations.

Characterization of Algebraic Thinking

A second analysis was made of the students' responses for the algebraic form of each problem in order to draw inferences about the nature of the students' algebraic thinking. This analysis focused on noting the appearance of algebraic symbols in formulas and equations and examining the uses made of these symbols. Student papers were separated into three categories; (A) algebraic symbols were used to represent equations expressing the mathematical relationships unique to the given problem, (B) algebraic symbols were used to represent familiar formulas, and (C) no literals were used - all symbols in the response were numerical. Table 2 presents the distribution of algebraic categories across the three solution strategies developed in the first analysis.

The nature of the mathematical relationships inherent in each problem situation influenced the types of possible responses. In the geometric problem many of the incorrect responses used irrelevant trigonometric formulas. Such responses were classified in category B. Since the relationships in the income tax problem cannot be expressed by any classic formula, responses to this problem tended to be classified in categories A or C. The responses in category 3-B were those for which students used an irrelevant formula.

Table 2 Form of Algebraic Response for each Problem Solving Category

	Geometric Figure (algebraic form)				Income Tax (algebraic form)			
Category	A	B	C	Total	A	B	C	Total
1	10 100%	0 0%	0 0%	10	18 90%	0 0%	2 10%	20
2	7 35%	12 60%	1 5%	20	14 87.5%	0 0%	2 12.5%	16
3	0 0%	17 89.5%	2 10.5%	19	3 21.4%	5 35.7%	6 42.9%	14

In category A students represented the mathematical relationships of the problem situation in the form of an equation in which two separate algebraic expressions containing a single variable were set equal to each other. The solution to the problem was obtained through algebraic manipulations of this equation rather than through the series of numerical computations characteristic of categories B and C. Category A represents an algebraic approach to the problem while categories B and C are essentially arithmetic.

Algebraic versus Arithmetic Thinking

Sfard and Linchevski (1994) propose a model of the development of algebra which, among other aspects, addresses the distinction between an arithmetic and an algebraic use of symbols. Based on an epistemological analysis supported by the historical development of the subject, this model presents the development of algebra as a sequence of two hierarchal levels - generalized arithmetic and abstract algebra. Within each level the perception of algebraic entities passes through an operational and then a structural stage related to the process-object duality inherent in many mathematical objects.

The categories described in the present study are found in this model within the operational and structural stages of the first level (generalized arithmetic perceptions of algebra). (Category C represents a purely arithmetic level which precedes algebra.) Category B corresponds to what Sfard and Linchevski call a numeric focus within the generalized arithmetic level's operational stage.

According to the model, students at this phase of development perceive algebraic formulas as vehicles that convey a sequence of numerical processes of which the end result is a number. The focus is on executing rather than on expressing, manipulating, and studying these processes. Student responses in category A fit a focus within the generalized arithmetic level's structural stage in which the perceptions of the structure of algebra involve variables denoting unknown but fixed values.

The nature of the problems did not make it possible to determine if students in category A also possessed a more abstract functional view of algebra represented in the model by a more advanced focus within the structural stage of generalized arithmetic. From this advanced perception, variables are viewed as representing changing (arbitrary) rather than constant magnitudes, and algebraic expressions (as opposed to numbers) are perceived as representations of the final result.

The findings of the present study support the conclusions reached by Sfard and Linchevski that the algebraic understanding of students is not very advanced by the end of their high school studies. In the present study 47 of the 99 students were classified in categories B and C, which placed them at or below the model's lowest level of algebraic thinking. Out of the remaining 52 students, only 28 were placed in category A-1 which implied an ability to reason successfully using a structural approach to algebra.

These findings do not bode well for the students' future mathematics achievement. It may be difficult to advance their level of algebraic thinking based on their previous years of traditional school mathematics. Sfard and Linchevski (p. 223) conclude that "since the functional approach is not easily accessible even for better students, it is not so difficult to understand why the mechanistic, pseudostructural approach may eventually dominate student's thinking like an overgrown weed, leaving no room for other, more meaningful perspectives."

It may be that a solution to the poor algebraic preparation of entering college students will require radical changes in the way that algebra is structured in the K-12 school curriculum. Kaput (1994) suggests a strand approach to curriculum design in which "big mathematical ideas"[1] are systematically developed throughout the complete school experience. Kaput also proposes that algebra be introduced in parallel with arithmetic as a natural outgrowth of quantitative reasoning. In this type of reasoning relationships among quantities rather than numbers and numeric relationships are of interest (Thompson, 1994). The model of Sfard and Linchevski with its stages of alternating operational and structural thought may provide the framework on which to build such an algebraic strand.

Even though the problems used in this study are similar to those encountered in traditional school algebra, their use in reformed curricula should not be rejected. Simon (1994, p. 89) points out that "activities that were used previously to "practice" a new idea may be used as situations for exploration which can contribute to the development of the idea." Problems similar to those in the study were used in the precalculus class to focus the students' attention on the differences between arithmetic and algebraic thinking.

References

Kaput, J. (1994). Long term algebra reform: democratizing access to big ideas. Draft form of unpublished manuscript.

Kieran, C. (1992). The learning and teaching of school algebra. in D. A. Grouws (Ed.) *Handbook of research on mathematics teaching and learning.* (pp. 390-419). New York: MacMillan Publishing Company.

Sfard, A. & Linchevski, I. (1994). The gains and pitfalls of reification - the case of algebra. *Educational Studies in Mathematics 26*, 191-228.

Simon, M. A. (1994). Learning mathematics and learning to teach: learning cycles in mathematics teacher education. *Educational Studies in Mathematics 26*, 71-94.

Simon, M. A. & Stimpson, V. C. (1988). Developing algebraic representation using diagrams. In A. F. Coxford (Ed.), *The ideas of algebra, K-12*. (pp. 136-141). Reston, VA: National Council of Teachers of Mathematics.

Steen, L. A. (1990). (ed.) *On the shoulders of giants: new approaches to numeracy*. Washington, D. C.: National Academy Press.

Thompson, P. W. (in press). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*.

---

[1]Steen (1990) suggests the following themes as strands around which mathematics curriculum can be organized: Quantity, Shape, Uncertainty, Dimension, and Change.