PRECALCULUS

Chapters 8 and 9

Warren W. Esty

CHAPTER 9

Conic Sections

Section 9.1. Conic Sections: Parabolas

Curves called conic sections have been extensively studied for over two thousand years. They include parabolas, circles, ellipses, and hyperbolas (Figures 1A, B, and C).

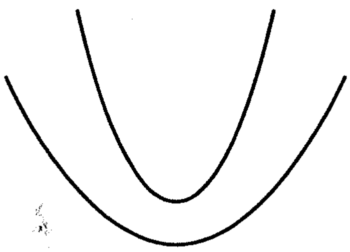


Figure 1A: Two parabolas.

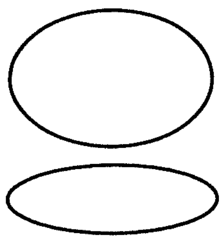


Figure 1B: Two ellipses

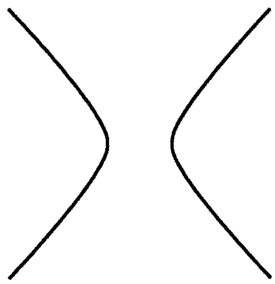


Figure 1C: One hyperbola.

By definition, a <u>conic section</u> is a curve formed as the intersection of a cone and a plane (Figures 2A, B, and C).

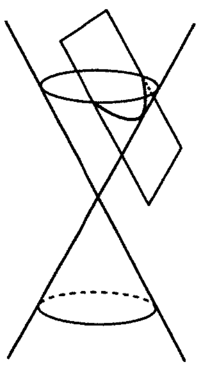


Figure 2A: A parabola as a cross section of a cone.

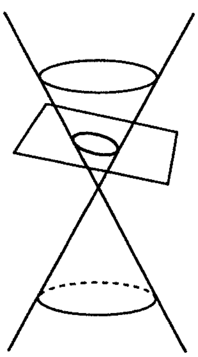


Figure 2B: An ellipse as a cross section of a cone.

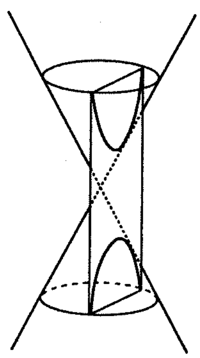


Figure 2C: A hyperbola as a section of a cone.

Different types of curves are formed depending upon the angle at which the cone is cut. Parabolas result when the plane is parallel to the edge of the cone (Figure 2A). Ellipses result when the plane is tilted to cut only one part of the cone (Figure 2B). Hyperbolas result when the plane is tilted to cut both parts of the cone (Figure 2C).

Conic sections are not merely mathematical curiosities; they are important shapes of nature. The Polish astronomer Kepler (1571-1630) discovered that planets move in ellipses about the sun. Man-made satellites and some comets also move in elliptical orbits, and some comets move in orbits that form one branch of a hyperbola. Disregarding friction, projectiles move in parabolic paths. To explain why objects move in these orbits takes calculus and Newton's law of gravity.

Other applications concern vision and light. In the real world circles are extremely common. Viewed from an angle, a circle appears elliptical; photographs and paintings illustrate circles as ellipses. The tip of a shadow of a fixed object (for example, the tip of a shadow of a pole) describes one branch of a hyperbola as the day passes (problem B16).

Parabolic, hyperbolic, and elliptical shapes all have remarkable reflective properties. Satellite dishes, microwave relay towers, telescopes, solar energy collectors and parabolic microphones utilize parabolic shapes.

By 200 BC the Greeks had, as a purely intellectual exercise, discovered numerous fascinating properties of conic sections, including the reflective properties. Ancient results about conic sections are strong support for the argument that pure research may lead to important discoveries, even if no application is in mind when the research is undertaken. Centuries passed before the important practical applications of conic sections were devised.

After straight lines, the next most important curves are these conic sections. There are alternative ways to define these curves. We will not use the ancient "conic section" approach. Instead, we will define these curves using two-dimensional geometric definitions well-suited to algebraic geometry.

Parabolas. We begin with a description of parabolas in geometric terms and then translate that definition into algebraic notation.

Definition 9.1.1: Let L be a line in the plane and F be a point in the plane not on that line. The set of points that are equidistant from L and F is called a parabola (Figure 3).

The line L is called the directrix ("Dih REK tricks") of the parabola and the point F is called the focus ("FOH kus") of the parabola.

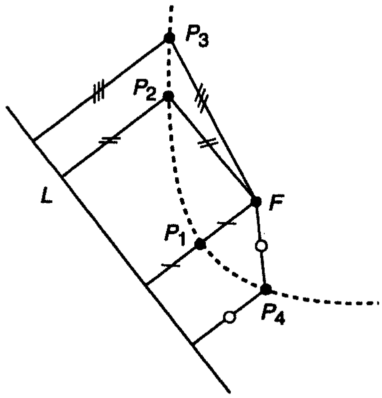


Figure 3: A line L, a point F, and a few points equidistant from both.

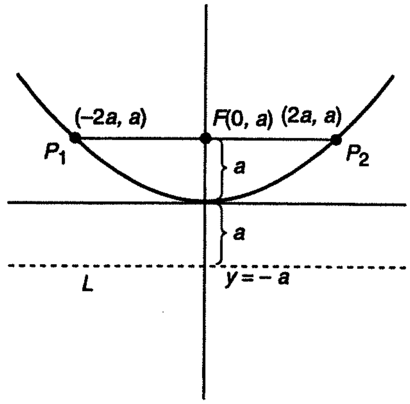


Figure 4: The axis-system pictured vertically.

If we want to express that geometric definition algebraically, we may. For that we need an axis system. It is most convenient to locate the axis system with the y-axis through the focus and perpendicular to the directrix, and the origin halfway (equidistant) between the focus and directrix. Figure 4 pictures the axis-system vertically as usual.

There are only two other points on the parabola other than the origin that are easy to find by inspection. The points ($-2a$, a) [P_1 in Figure 4] and ($2a$, a) [P_2] are also on the parabola because they are the same distance from the directrix and the focus. They are $2a$ units above the line (measured perpendicular to the line) and $2a$ units directly to the side of the focus. The line segment through the focus connecting P_1 to P_2 is called the *latus rectum* (that is a Latin term) and serves as a description of how wide the parabola is. The latus rectum is $4a$ units long -- exactly twice the distance from focus to directrix.

The other points on the parabola are not so simple to discover. However, using algebra, we may treat the geometric description of a parabola as a word problem, build our own formulas, and then set up the equation.

<u>Algebraic Description of a Parabola</u>. To describe the geometric definition in 9.1.1 algebraically we need to express the two distances mentioned. Then we will set the distances equal to each other.

Let a point be denoted by (x, y). How far is it from the focus (0, a)?

By the distance formula 3.3.2, the distance from (x, y) to (0, a) is (Figure 5)

$$\sqrt{(x-0)^2+(y-a)} \;=\; \sqrt{x^2+y^2-2ay+a^2}.$$

Points (x, y) on the parabola will be in the top half of the plane (Figure 5), so the distance from (x, y) to the directrix $y = -a$ is simply $y + a$.

Now, set these two distances equal and simplify.

$$y + a = \sqrt{x^2 + y^2 - 2ay + a^2}.$$

Squaring both sides,

$$(y + a)^2 = x^2 + y^2 - 2ay + a^2.$$
$$y^2 + 2ay + a^2 = x^2 + y^2 - 2ay + a^2.$$

Simplifying,

(9.1.2A) $$4ay = x^2.$$

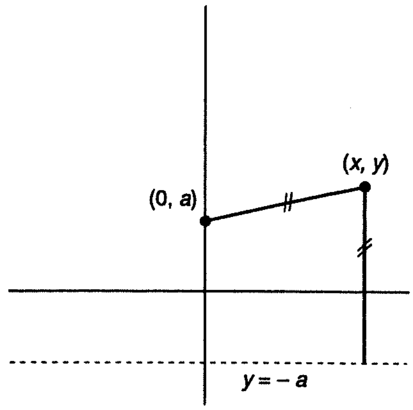


Figure 5: Focus $(0, a)$, directrix $y = -a$, and a point (x, y) on the parabola.

Dividing by $4a$, we can isolate y.

Vertical Parabola Centered at the Origin. An equation of a parabola with focus $(0, a)$ and directrix $y = -a$ is

(9.1.2B) $$y = \frac{x^2}{4a}.$$

The parabola goes through the origin and the focus and directrix are $|2a|$ units apart.

If a is positive, the parabola "opens upward," as parabolas usually do in pictures, for example, Figures 1A and 4. If a is negative, the focus is below the directrix and the parabola "opens downward" (Figure 9).

Parabola

$y = x^2/(4a)$

focus: $(0, a)$
directrix: $y = -a$
vertex: $(0, 0)$

distance from focus to directrix: $2a$

Example 1: The best-known algebraic description of a parabola is "$y = x^2$." Locate its focus and directrix.

In 9.1.2B, "a" is a parameter which locates the focus and directrix. We are given that the parabola is "$y = x^2$." In 9.1.2B this is "$y = x^2/(4a)$." So,

$$y = x^2 = x^2/(4a).$$

Solving for a, $4a = 1$ and $a = 1/4$. Therefore, the focus is $(0, 1/4)$ and the directrix is $y = -1/4$.

Example 2: Find an equation of the parabola with focus (0, 1) and directrix $y = -1$.

Half way between the line and the focus is the vertex. Here, that point is the origin, so equation 9.1.2 fits. Since the distance between the focus and the origin is $2a$, which is here 2 units, $a = 1$. Plugging into Equation 9.1.2B, the equation is

$$y = x^2/4.$$

<u>**Location Shifts**</u>. If a parabola is defined geometrically on a "blank page" *before* the axis system is laid down, we can choose our axis system to fit the parabola as we did in Figure 4. Then the algebra is relatively easy. The resulting formula is 9.1.2B, which has only one parameter, a.

If the directrix and focus are given on a pre-existing axis system, the algebra is more complicated. Nevertheless, any parabola can be expressed algebraically in terms of location shifts, scale changes, or rotations of this basic parabola. Location shifts and scale changes are dealt with here exactly as in Section 2.2 on composition of functions and the part of Section 3.3 on circles and ellipses.

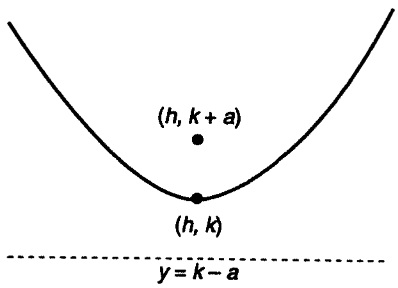


Figure 6: A vertical parabola that "opens upward" with vertex (h, k).

First consider parabolas where the axis of symmetry is still vertical, but the location is different. It is easy to algebraically shift the vertex to a new location (Figure 6). For example, to locate the vertex at (h, k) ["h" and "k" are the traditional letters used to locate the vertex], we can substitute "$x - h$" for "x" and "$y - k$" for "y" in 9.1.2. The result is immediate.

$$y - k = \frac{(x-h)^2}{4a}$$

<u>Vertical Parabolas</u>. The equation of the standard geometric form of a parabola with vertical axis and vertex at (h, k) is

(9.1.3) $$y = \frac{(x-h)^2}{4a} + k.$$

The directrix is $y = k - a$ and the focus is $(h, k + a)$ (Figure 6).

Example 3: Find the algebraic representation of a parabola with focus at (5, 7) and directrix $y = 1$.

Draw a picture (Figure 7).

Since the directrix is horizontal, the axis of symmetry will be vertical and the standard geometric form in 9.1.3 fits. The distance from the directrix $y = 1$ to the focus (5, 7) is 6. This distance is $2a$, so $a = 3$. The vertex is half way in between the focus and directrix, so the vertex is (5, 4). Therefore, the equation is

$$y = (x - 5)^2/12 + 4$$

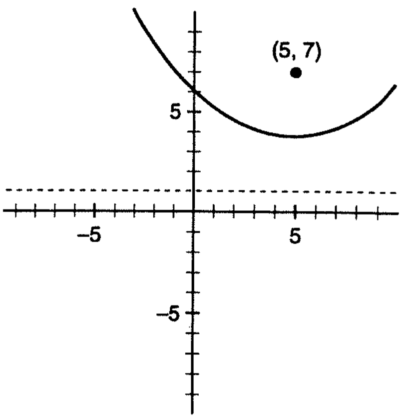


Figure 7: A parabola with focus (5, 7) and directrix $y = 1$. [-10, 10] by [-10, 10].

Example 4: Find the focus and directrix of the parabola with equation "$y = x^2/8 + x$."

Graph it (Figure 8). You should be able to approximate the answers just by looking at the graph.

This equation can be manipulated into the form in 9.1.3, which exhibits the three parameters h, k, and a. Complete the square (as in Section 3.2) to obtain

$$\begin{aligned} y = x^2/8 + x &= (1/8)(x^2 + 8x) \\ &= (1/8)(x^2 + 8x + 16 - 16) \\ &= (1/8)(x + 4)^2 - 2 \end{aligned}$$

The algebraic form which exhibits all three parameters would be

$$y = \frac{(x - -4)^2}{4(2)} + -2.$$

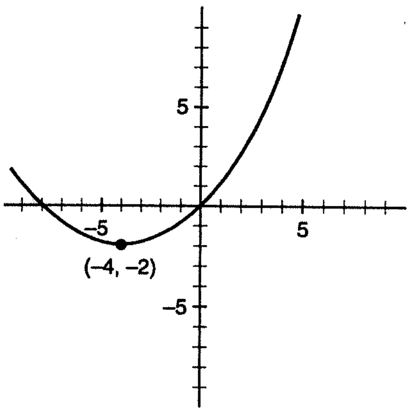


Figure 8: $y = x^2/8 + x$. [-10, 10] by [-10, 10].

So $h = -4$, $k = -2$, and $a = 2$. The focus is 2 units above the vertex and the directrix is horizontal two units below the vertex. Vertex: (-4, -2). Focus: (-4, 0). Directrix: $y = -2 - 2 = -4$.

A vertical parabola is upside down ("opens downward") if and only if a is negative.

Example 5: A parabola has vertex (4, 5) and focus (4, 0) (Figure 9). Find its equation.

The axis of symmetry is vertical, so 9.1.3 fits. The directed distance from the vertex to the focus

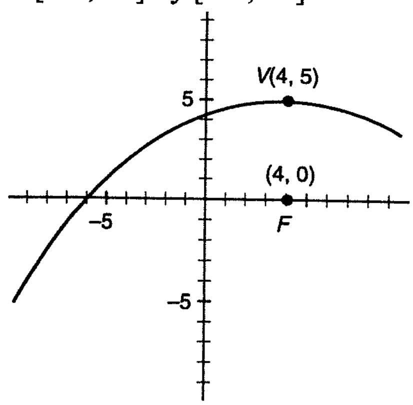


Figure 9: A parabola with vertex (4, 5) and focus (4, 0). [-10, 10] by [-10, 10].

is -5 (negative, because the focus is *below* the vertex) so $a = -5$. The center (4, 5) exhibits h and k. The equation is

$$y = -(x - 4)^2/20 + 5.$$

<u>Horizontal Parabolas</u>. The difference between a horizontal graph and a vertical graph is the difference between "x" and "y". Switch the letters and the role of the axes switches, so what was vertical becomes horizontal, and vice versa.

An equation of the standard geometric form of a parabola with horizontal axis and vertex at (h, k) is (switching "x" and "y" in 9.1.3):

(9.1.4) $$x = \frac{(y-k)^2}{4a} + h.$$

The directrix is $x = h - a$ and the focus is $(h + a, k)$ (Figure 10).

Solving 9.1.4 for y,

(9.1.5) $$y = \pm\sqrt{4a(x-h)} + k.$$

This is the functional form (with two functions) suitable for graphing horizontal parabolas with a graphics calculator.

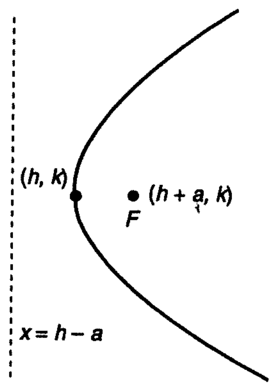


Figure 10: A horizontal parabola.
Vertex: (h, k). Focus $(h + a, k)$
Directrix: $x = h - a$

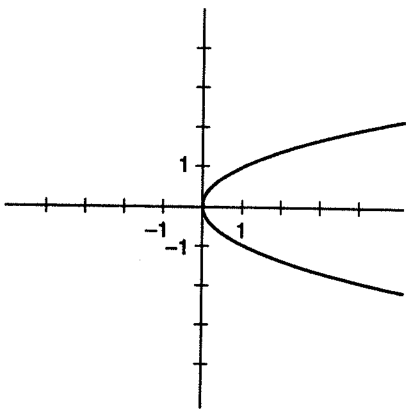


Figure 11: $x = y^2$.
[-5, 5] by [-5, 5].

Example 6: The best-known horizontal parabola is given by $x = y^2$ (Figure 11). We have seen the top half graphed many times as $y = \sqrt{x}$. Locate its focus and directrix.

This is just Example 1 with the letters switched. In Example 1 the focus of "$y = x^2$" was (0, 1/4). Here the roles are reversed; the focus is (1/4, 0). In Example 1 the directrix of "$y = x^2$" was "$y = -1/4$." Here the roles are reversed. The directrix of "$x = y^2$" is "$x = -1/4$."

If you want to use 9.1.4 or 9.1.5 to do Example 6, "$x = y^2$," from scratch, you may. We are given $x = y^2$, which, by 9.1.4, is also $x = (y - k)^2/(4a) + h$. Comparing the two expressions, we see $k = 0$, $a = 1/4$, and $h = 0$.

Reflections. Parabolas have an important reflective property. Narrow-beam lights (such as some flash lights and search light beams) have a mirrored reflective surface behind the light source (bulb filament) so that light reflects straight forward (Figure 12). The shape that does this is a parabola. If the inner surface of a parabola is mirrored, light from the focus is reflected parallel to the axis of symmetry, regardless of where it hits the mirror.

Solar energy collectors, satellite dishes, and parabolic microphones also use this property, but in reverse. Parallel light rays, electromagnetic waves, or sound waves coming in toward the parabolic dish are reflected toward a single point, the focus (Figure 12).

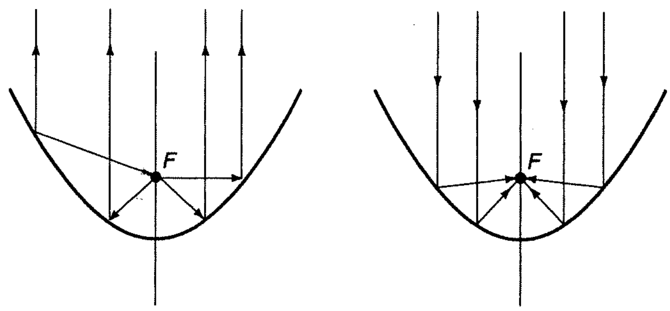


Figure 12: Reflections from a parabola.

This property can be discussed algebraically using a principle expressed by Fermat in 1650:

(9.1.6) "Light traveling from one point to another will follow a path such that, compared to other nearby paths, the time required will be a minimum or a maximum, or will remain unchanged."

Most examples require the time to be a minimum (Problem B14). In physics, this principle can be used to explain why light bends ("refracts") as it passes through the boundary between air and water, because light travels more slowly through water (Problem B13). Here we will use it to illustrate that light from the focus of a parabola reflects from the inner side of the parabola parallel to the axis of symmetry.

Example 7: This example would work with any parabola, but, for convenience, consider the famous parabola $y = x^2$ from Example 1. Imagine the inner surface to be reflective and the focus to be a light source (like the filament of a light). From Example 1, the focus is $F = (0, 1/4)$. Let A be a point above the parabola (Figure 13). Suppose a light ray from the focus reflects off the parabola and passes through point A. Where does that ray reflect off of the parabola?

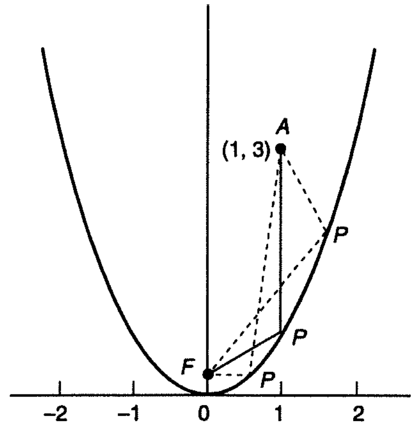


Figure 13: Light from the focus reflects off the parabola at P and goes through point A. Where is point P? [-2.5, 2.5] by [0, 5].

By the Fermat Principle, we need to find the point P on the parabola such that the total travel time from F to P to A is minimum. This is the same as finding the point P on the parabola such that the total distance from F to P to A is minimum, since light travels at a constant speed (Figure 13).

For this we must build a formula for the total distance, depending upon P. Then we can minimize the distance, perhaps graphically,

To simplify matters let's use a particular point A, say (1, 3). We need the distances from F to P and from P to A. Suppose we denote P by (x, y). Using the distance formula 3.1.11,

$$d(F,P) = \sqrt{(x-0)^2 + (y-1/4)^2}\,.$$
$$d(P,A) = \sqrt{(x-1)^2 + (y-3)^2}\,.$$

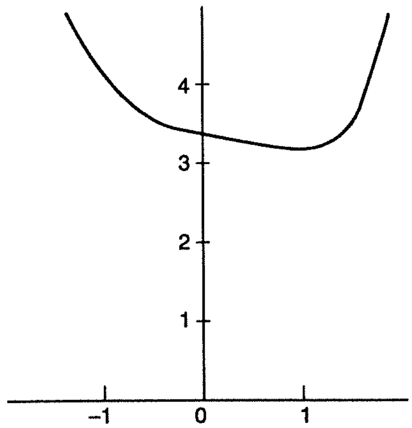


Figure 14: The distance traveled from F to P to A, in terms of the x-value of P. [-2, 2] by [0, 5].

These are valid for any point (x, y), but the problem has a constraint (recall Section 3.6 on constraints). For a given x, we cannot use just any y; y must be on the parabola. For any x, y must be x^2. Substituting x^2 for y and adding,

$$\text{total distance } = \sqrt{(x-0)^2 + (x^2-1/4)^2} + \sqrt{(x-1)^2 + (x^2-3)^2}\,.$$

Graph this distance (Figure 14).

When the total distance is a minimum, $x = 1$, which is the "1" from the point $A = (1, 3)$. That is, this reflected light ray is reflected parallel to the axis of symmetry.

In general (and we can prove this using calculus), regardless of the location of

A, the point P on the curve that minimizes the total reflected distance from the focus always has the same x-value as A itself.

Conclusion. Conic sections are important. In addition to being cross sections of a cone cut by a plane, they can be described geometrically in terms of distance. Also, they have relatively simple algebraic equations with parameters that exhibit the center and scale if the axis system is laid out parallel to the axis of symmetry.

Terms: conic section, parabola, focus, directrix.

Exercises for Section 9.1, "Conic Sections: Parabolas":

^^^^ Rewrite the equation in "standard geometric form." Identify the vertex, axis of symmetry, focus, and directrix of each parabola. Then roughly sketch it.

A1. $y = x^2$. [F: (0, 1/4)]
A2. $y = 2x^2$. [dir: $y = -1/4$]
A3. $y = (x - 1)^2 + 2$. [F: (1, 9/4)]
A4. $y = (x + 3)^2 + 1$. [V: (-3, 1)]
A5. $y = x^2 + 4x$. [dir: $y = -17/4$]
A6. $y = x^2 - 6x$. [axis: $x = 3$]
A7. $y = 2 - x^2$. [V: (0, 2)]
A8. $y = -(x/3)^2$. [F: (0, -9/4)]
A9. $y = 2x^2 - 8x - 5$. [axis: $x = 2$]
A10. $y = 4x^2 + 12x + 1$. [V: (-3/2, -8)]
A11. $x = y^2$. [dir: $x = -1/4$]
A12. $x = y^2/8$. [F: (0, 2)]
A13. $x = (y - 2)^2 + 3$. [V: (3, 2)]
A14. $x = (y - 7)^2 + 4$. [F: (17/4, 7)]
A15. $x^2 + 2x - y = 5$. [dir: $y = -25/4$]
A16. $2x + y^2 = 6y$. [V: (9/2, 3)]

^^^^ Find the equation of the parabola with the given properties.

A17. Vertex (0, 0), focus (0, 1).
A18. Vertex (0, 0), focus (0, -2).
A19. Vertex (0, 0), focus (1, 0).
A20. Vertex (0, 0), focus (-3, 0).
A21. Vertex (3, 4), focus (3, 10).
A22. Vertex (0, 2), focus (0, -4).
A23. Vertex (0, 0), directrix $y = -2$.
A24. Vertex (0, 0), directrix $x = -1$.
A25. Focus (1, 2), directrix $y = 0$.
A26. Focus (1, 2), directrix $x = 0$.
A27. Focus (2, 0), directrix $y = -10$.
A28. Focus (0, 1), directrix $x = -2$.
A29. Vertex (0, 0), axis $y = 0$, through (4, 1).
A30. Vertex (0, 0), axis $x = 0$, through (2, 10).

^^^^^^^^

B1.* a) Define *parabola*. b) Illustrate the definition with a sketch.

B2.* Sketch $y = (x - h)^2/(4a) + k$, labeling the vertex, axis of symmetry, focus, and directrix.

B3. A solar reflector is parabolic, 10 feet wide and 2 feet deep. Where is its focus?

B4. A parabolic microphone is 2 feet wide and 4 inches deep. Where is its focus?

B5. A parabolic searchlight is 1 foot wide and 6 inches deep. Where is its focus?

B6. A parabolic solar reflector is 6 inches deep and has the focus 2 feet in front of the vertex. How wide is it?

B7. A parabolic solar reflector is 3 inches deep and has the focus 18 inches in front of the vertex. How wide is it?

B8. Suppose sound travels about 1100 feet per second. Location A is 500 feet from location B. Suppose a burst of sound is recorded at location A 1/10 second before it is recorded at location B. Draw a sketch, locate an axis system on the sketch, and find the algebraic equation satisfied by the possible locations of the origin of the burst of sound. (Do not bother to simplify it.)

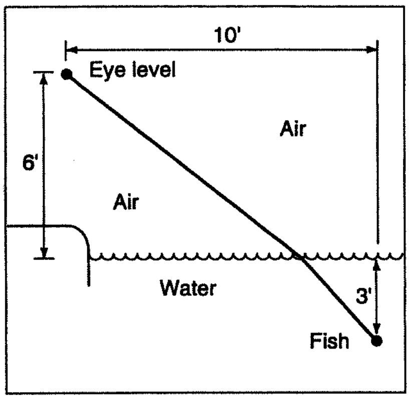
B9. Suppose sound travels about 1100 feet per second. Location A is 2200 feet from location B. Suppose a burst of sound is generated at location A, reflects off of point P, and is recorded at location B 3 seconds later. Draw a sketch, locate an axis system on the sketch, and find the algebraic equation satisfied by the possible locations of point P. (Do not bother to simplify it.)

B10. Let $y = x^2/4$ and P be a point on that curve. Let F be the point (0, 1), and B be the point (2, 5).
a) Find the expression for the sum of the distances from B to P and from P to F.
b) Use a graph to find the x-value of the point on the curve that minimizes that expression.

B11. Let $y = x^2/100$ and P be a point on that curve. Let F be the point (0, 25), and B be the point (5, 20). a) Find up the expression for the sum of the distances from B to P and from P to F.
b) Use a graph to find the x-value of the point on the curve that minimizes that expression.

B12. Draw a picture to illustrate why the distance from (x, y) to the line $y = -a$ is $|y + a|$.

B13. A person with eye level 6 feet above the water sees an object that is actually 3 feet below the water and 10 feet away horizontally (see the figure). Light from the object reaching the person's eye will not travel in a straight line, because light travels more slowly in water. It takes 1.33 times as long for light to travel 1 foot under water as in air. a) Find the path such that light from the object to the person's eye travels the minimum time (according to Fermat's Principle, 9.1.6). Describe the path in terms of the horizontal distance from the eye to where the light leaves the water. b) The object appears to be not as deep as it really is. Why?



#B14. Think of an example where a reflection follows the Fermat Principle and yields maximum time (instead of minimum time).

#B15. Think of an example where a reflection follows the Fermat Principle and yields constant time (instead of minimum time).

B16. Use the idea of a "conic section" to explain why the shadow of the tip of a pointed post traces out one branch of a hyperbola on the ground as the day passes. [Hint: Where is the cone? What makes the section?]

Section 9.2. Ellipses and Hyperbolas

Ellipses and hyperbolas are conic sections that can be described with geometric definitions using distances from two fixed points. If the *sum* of the distances from any point on the curve to the two fixed points is a constant, the curve is an ellipse. If the *difference* is a constant, the curve is a hyperbola.

Definition 9.2.1. An ellipse is the set of all points in the plane such that the sum of the distances from two fixed points is a constant. Each of the fixed points is called a focus (pronounced "FOH kus"; the plural is "foci," pronounced "FOH sigh").

Figure 1 illustrates two fixed points, F_1 and F_2, and a curve of points P such that the sum of the lengths of the line segments from F_1 to P to F_2 is a constant. Using this definition, an ellipse can be drawn using a loop of string around two pins (serving as the foci). Use the point of a pencil to pull the string taut in any direction, forming a triangle with the point as the third vertex (Figure 2). That point will be on the ellipse, which will be drawn as the point moves through all positions at the extreme end of the triangular loop. I have drawn many ellipses this way and it works well.

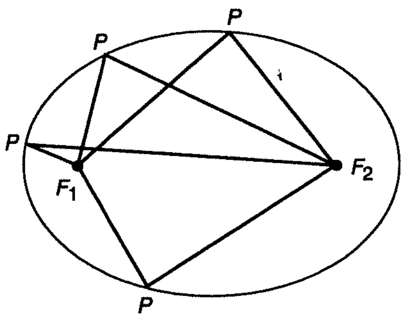


Figure 1: An ellipse. P varies, but the total distance from F_1 to P to F_2 is constant.

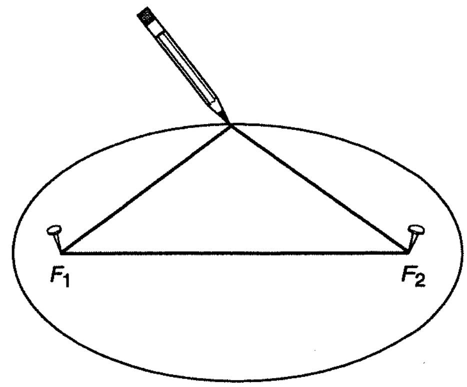


Figure 2: A way to draw an ellipse.

A circle is a special case of an ellipse when the two foci are the same point (the center). Then the distance from the center to any point on the circle and back to the center is a constant (twice the radius).

Applications. There are many applications of definitions based on distance. Some arise in the use of location-finding technology such as sonar, radar, and the Global Positioning System.

Example 1: Imagine we create a burst of sound at location F_1 and record when the sound arrives at location F_2 after it reflects off of a surface at an unknown location P. The sound travels from F_1 to P to F_2 (Figure 3). By keeping track of when the burst is generated at F_1 and when it is received at F_2 we can tell how long the sound travels. The total distance from F_1 to P to F_2 is determined by the time the sound travels (using "distance equals rate times time," because we know the speed of sound). Knowing only the *total* distance and the locations of F_1 and F_2, the position of P is not determined precisely, but it must be somewhere on a particular ellipse, according to the definition of *ellipse* in 9.2.1 (Figure 3).

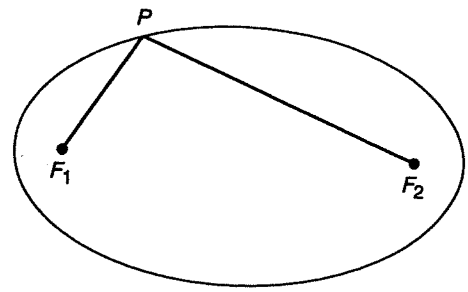


Figure 3: When the total distance from F_1 to P to F_2 is known, point P is on an ellipse with foci at F_1 and F_2.

Example 1, continued: So far, the only listening point is F_2. Now suppose we also listen at F_1 for the echo from P. The distance from F_1 to P and back to F_1 is determined by the travel time of the sound. So the distance from F_1 to P will be known. Then P must be on a circle with known radius centered at point F_1 (the circle in Figure 4). So, if we listen at both F_1 and F_2, then, P is on both the circle and the ellipse. Therefore the location of P almost determined; it is one of two points where the circle and ellipse intersect (Figure 4).

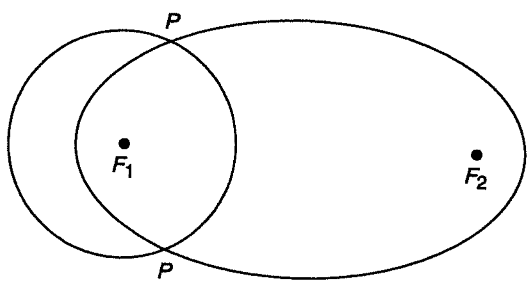


Figure 4: Point P is on an ellipse with foci at F_1 and F_2 and the total distance F_1 to P to F_2 is known. P is also on a circle with known radius centered at F_1. Therefore the location of P almost determined; it is one of two points.

When the burst is produced by the listener as in Example 1, the location-finding method is said to be "active." When the object at P itself generates the burst, there is a similar "passive" method of determining location. The object can be located on a hyperbola.

Definition 9.2.2. A hyperbola (high PER boh lah) is the set of all points in the plane such that the (absolute value of the) difference of the distances from two fixed points is a constant (Figure 5). Each of the fixed points is called a focus of the hyperbola.

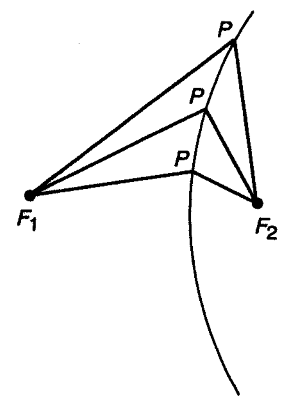


Figure 5: One branch of a hyperbola. P varies, but the difference between PF_1 and PF_2 is constant.

A hyperbola has two branches. Each consists of points closer to the focus inside that branch.

We continue our discussion of finding locations.

Example 2: Again, suppose two listening locations, F_1 and F_2, are fixed. Imagine a burst of sound originating at unknown location P at an unknown time (Figure 5). Can we use the arrival times of the sound at F_1 and F_2 to locate the point of origin?

If the sound arrives at both locations at the same instant, then the source P is equidistant from each. This is a special case resulting in a line of possible locations (Figure 6). So a line is a "degenerate" special case of a hyperbola. Usually we use the term "hyperbola" to apply to the other cases where the difference in distances is not zero.

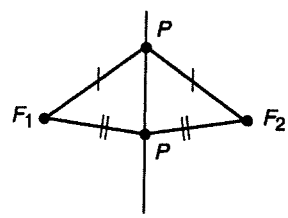


Figure 6: A line of points equidistant from F_1 and F_2.

Suppose the sound arrives at F_2 before it arrives at F_1, so the source is closer to F_2 (Figure 5). The difference in arrival times can be used to compute the difference between the distances from P to F_1 and from P to F_2. According to the definition of a hyperbola, that constant difference locates P somewhere on a particular hyperbola with F_1 and F_2 as foci, and on the branch closer to F_2.

This result is like knowing that an unknown point is on a particular ellipse–it is not nearly enough to locate the point. In the previous example, when the sound was generated at the listening point, we needed a second listening point to almost determine the location of the reflecting object. Now, when the sound is generated at an unknown point source, we need a third listening point to almost determine the location of the source. Each pair (three points make three pairs of points) will locate the point on one branch of a hyperbola. The three hyperbolas are likely to intersect in only one point. Figure 7 illustrates branches of two hyperbolas on which the point P must lie.

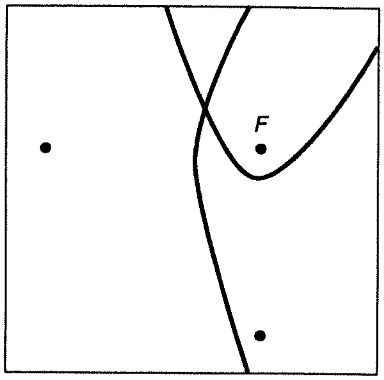


Figure 7: A point at the intersection of branches of two hyperbolas with focus F in common. The other 2 foci are marked.

This location-finding method has applications to locating submarines (using Sonar) and airplanes using radar waves (instead of sound waves). A method which is similar in spirit to this example is used by the Global Positioning System (GPS) to precisely determine locations on earth. In that case several satellites transmit signals simultaneously and they are received at the GPS device at different times because the GPS device is at different distances from the different satellites. Because radio waves travel so extremely rapidly (186,000 miles per second), it takes an extremely precise clock to detect the differences in arrival times. These time differences are then converted to differences in distances from the satellites, which then yield equations similar in spirit to those in 9.2.4 below, which are then solved to yield the location of the device.

Algebraic Descriptions of Ellipses and Hyperbolas. The geometric definitions of *ellipse* and *hyperbola* mention distances from two foci. Therefore, to describe them algebraically, we need to express both distances. For an ellipse, their *sum* is set equal to a constant. For a hyperbola, their *difference* is set equal to a constant.

It is convenient to center the ellipse or hyperbola at (0, 0) and have the x-axis down the middle by locating the foci at $(-c, 0)$ and $(c, 0)$, for some $c \geq 0$.

Suppose we call a point in the plane (x, y). How far is it from the focus $(c, 0)$ (Figure 8)? The formula is

$$\sqrt{(x - c)^2 + y^2}\ .$$

How far is it from $(-c, 0)$? The formula is

$$\sqrt{(x + c)^2 + y^2}\ .$$

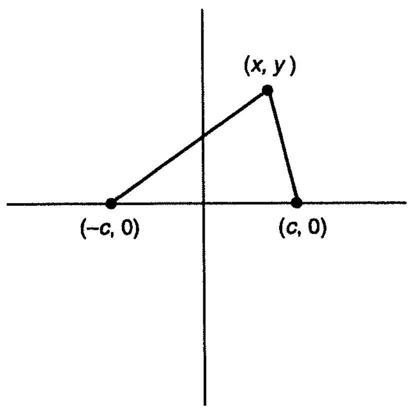


Figure 8: (x, y) and the two foci, $(c, 0)$ and $(-c, 0)$.

It is traditional to let the constant that is the sum or difference of these two distances be denoted by $2a$. For an ellipse, $2a$, the sum of the distances, must be greater than $2c$, which is the straight-line distance between the two foci. So, for an ellipse, $a > c$. For a branch of a hyperbola, the difference must be less than the distances between the two foci, so $2a$ must be less than $2c$ and $a < c$. The number $2a$ may be zero as a special case (a line, Figure 6). The equations are

(9.2.3, ellipse) $$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a\,.$$

(9.2.4, hyperbola) $$\sqrt{(x-c)^2+y^2}-\sqrt{(x+c)^2+y^2}=\pm 2a\,.$$

This hyperbola equation with the minus sign determines the branch closer to $(c, 0)$ than to $(-c, 0)$. With the plus sign it determines the branch closer to $(-c, 0)$.

Neither of these equations is pleasing. Fortunately, they can be simplified by squaring twice (by the procedure outlined in Example 4.3.8).

<u>Details</u>. We simplify the case of the ellipse here.

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a$$

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$

Squaring both sides

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{x^2-2cx+c^2+y^2}+x^2-2cx+c^2+y^2\,.$$

Several terms cancel.

$$4cx-4a^2=-4a\sqrt{x^2-2cx+c^2+y^2}$$

$$cx-a^2=-a\sqrt{x^2-2cx+c^2+y^2}$$

$$(cx-a^2)^2=a^2(x^2-2cx+c^2+y^2)$$
$$c^2x^2-2a^2cx+a^4=a^2x^2-2a^2cx+a^2c^2+a^2y^2$$

Now, two terms cancel. Putting all the x's on the right,

$$a^2(a^2-c^2)=(a^2-c^2)x^2+a^2y^2.$$

Now, because $a > c \ge 0$, $a^2 - c^2 > 0$. Thus we can let $b^2 = a^2 - c^2$.

$$a^2b^2=b^2x^2+a^2y^2.$$

Dividing through by a^2b^2, we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,.$$

This is the "standard form" of an ellipse (3.3.4) that we first encountered in Section 3.3 on distance, circles, and ellipses. There we recognized this equation as the equation of a circle with the scale changed. The graph of the unit circle is expanded by a factor of a in the x-direction and by a factor of b in the y-direction.

<u>Ellipses Centered at the Origin</u>. The "standard form" of ellipses centered at the origin is

(9.2.5) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .$$

The graph goes through $(a, 0)$, $(-a, 0)$, $(0, b)$, and $(0, -b)$. These four points make it easy to graph (Figures 9 and 10). The foci are c units from the origin, where $a^2 = b^2 + c^2$ if the major axis is horizontal and $b^2 = a^2 + c^2$ if the major axis is vertical.

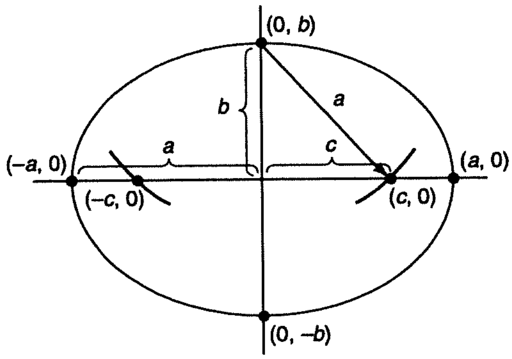


Figure 9: An ellipse, with $a > b$.
$x^2/a^2 + y^2/b^2 = 1$.
Foci at $(-c, 0)$ and $(c, 0)$.

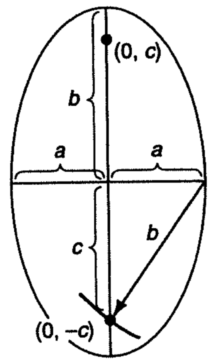


Figure 10: An ellipse with $b > a$.
$x^2/a^2 + y^2/b^2 = 1$.
Foci at $(0, c)$ and $(0, -c)$.

Ellipses are longest along the "major axis" through the foci. When $a > b$ as in Figure 9, the major axis is the horizontal line segment of length $2a$ from $(-a, 0)$ to $(a, 0)$. When $b > a$ as in Figure 10, the ellipse is taller than it is wide and the major axis is the vertical line segment of length $2b$ from $(0, -b)$ to $(0, b)$. Therefore, the greater of a and b is the length of the "semi-major" axis ("semi" means "half"). The distance from the foci to the center is denoted "c".

Through the center and perpendicular to the major axis is the "minor" axis. The smaller of a and b is the length of the "semi-minor axis" (half of the minor axis).

Some texts always use "a" for the length of the semi-major (longer) axis, regardless of whether "a" is associated with "x" or "y". This text always associates "a" with the x-direction and "b" with the y-direction.

If $a > b$ (Figure 9), $c^2 = a^2 - b^2$.
The length is horizontal.
The foci are on the horizontal axis of symmetry at $(-c, 0)$ and $(c, 0)$.

If $b > a$ (Figure 10), $c^2 = b^2 - a^2$.
The length is vertical.
The foci are on the vertical axis of symmetry at $(0, -c)$ and $(0, c)$.

In both cases, $c^2 = \text{larger}^2 - \text{smaller}^2$.

Given the equation in standard form, ellipses are easy to sketch (Figures 9 and 10).

Example 3: Sketch the ellipse and locate its foci given the equation

$$x^2/25 + y^2/9 = 1.$$

Because the form 9.2.5 fits, it is centered at the origin. $a = 5$ and $b = 3$, so it goes through (-5, 0) and (5, 0) on the major axis, which is the x-axis. The ellipse goes through (0, 3) and (0, -3) on the minor axis, which is the y-axis (Figure 11). The foci are on the major axis at $(-c, 0)$ and $(c, 0)$, where $a^2 - b^2 = c^2$, so $c = 4$. The foci are at (-4, 0) and (4, 0).

When $a > b$, as in this example, a is both the distance from the center to the horizontal ends and the distance from $(0, b)$ to the foci. Given the graph, but not the foci, the foci can be easily determined by striking an arc of length a from the top of the ellipse and noting the two points where it crosses the horizontal axis (Figure 9).

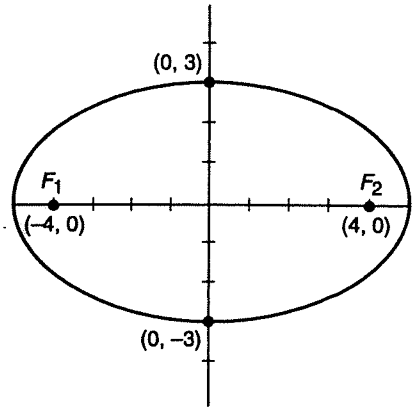


Figure 11: $x^2/25 + y^2/9 = 1$. [-5, 5] by [-5, 5].

Calculator Exercise 1: Learn how to graph ellipses on your calculator by solving for y in 9.2.5.

(9.2.6) $$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

You may need to graph two graphs, one for the upper half and one for the lower half. Use this approach to graph $x^2/25 + y^2/9 = 1$ on your calculator. Does your graph resemble Figure 11?

Example 4: Identify and sketch the ellipse with equation $3x^2 + 5y^2 = 40$.

The keys to "standard form" are that the right side is 1 and that the squared terms are alone on top with coefficient 1. Any multiple must be rewritten to be on the bottom.

$$3x^2 + 5y^2 = 40$$
$$3x^2/40 + 5y^2/40 = 1$$
$$\frac{x^2}{\left(\frac{40}{3}\right)} + \frac{y^2}{8} = 1.$$

$$\frac{x^2}{\left(\sqrt{\frac{40}{3}}\right)^2} + \frac{y^2}{\left(\sqrt{8}\right)^2} = 1.$$

This fits the form of an ellipse centered at the origin. From 9.2.5, $a = \sqrt{40/3} = 3.65$ and $b = \sqrt{8} = 2.82$. If we do not worry too much about the precise shape, the graph is easy to sketch using just the four points on the graph that the values of a and b provide (Figure 12).

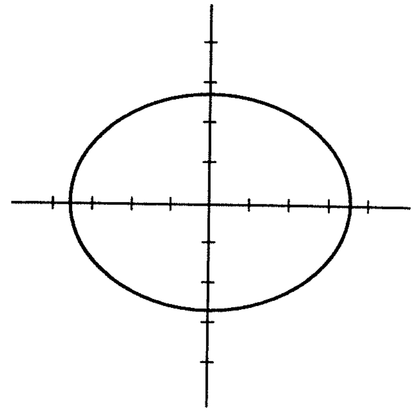


Figure 12: $3x^2 + 5y^2 = 40$. $a = 3.65$ and $b = 2.82$. [-5, 5] by [-5, 5].

Example 5: Identify the foci of the ellipse with equation

$$\frac{x^2}{25} + \frac{y^2}{169} = 1.$$

Here the major (longer) axis is the y-axis and therefore the foci lie on the y-axis at $(0, c)$ and $(0, -c)$, where, again, c^2 is determined as the larger denominator minus the smaller denominator. $c^2 = 169 - 25 = 144$, so $c = 12$. The foci are (0, 12) and (0, -12) (Figure 13). Be sure not to draw the ellipse pointed at the end. Even long and thin ellipses are really quite rounded at the ends.

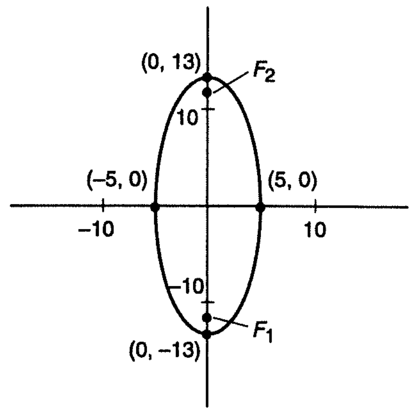


Figure 13: An ellipse with vertical major axis. $x^2/25 + y^2/169 = 1$. [-20, 20] by [-20, 20].

Location Shifts. To center an ellipse at (h, k), let "$x - h$" and "$y - k$" play the roles of "x" and "y" in 9.2.5 where the graph was centered at the origin.

The Standard Form of an Ellipse:

(9.2.7)
$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1.$$

This ellipse is centered at (h, k) and goes through the points $(h - a, k)$, $(h + a, k)$, $(h, k + b)$ and $(h, k - b)$ (Figure 14).

If $a > b$, the major axis lies along $y = k$ (in the x-direction), and the foci are c units from the center on the major axis at $(h - c, k)$ and $(h + c, k)$, where $c^2 = a^2 - b^2$.

If $b > a$, the major axis lies along $x = h$ (in the y-direction), and the foci are c units from the center on the major axis at $(h, k - c)$ and $(h, k + c)$, where $c^2 = b^2 - a^2$.

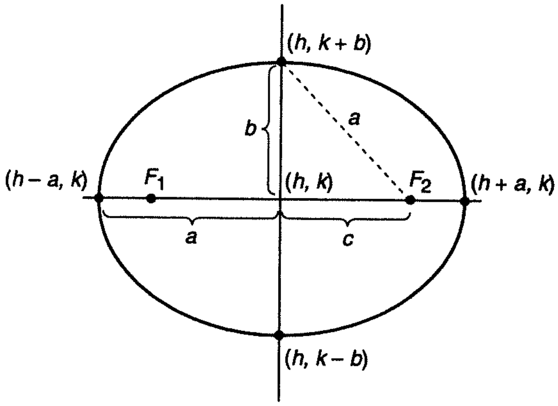


Figure 14: An ellipse with center (h, k), semi-major axis a, and semi-minor axis b.

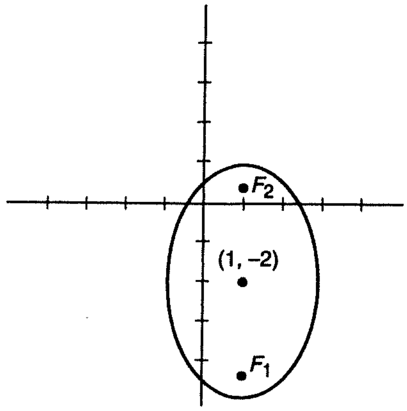


Figure 15: The ellipse with center (1, -2), $b = 3$ and $a = 2$. [-5, 5] by [-5, 5].

Example 6: Identify the ellipse with equation $9x^2 - 18x + 4y^2 + 16y = 11$.

Aim for the standard form of 9.2.7. If the coefficients on x^2 and y^2 were the same, it would be a circle and we would complete the square to find the center. Since the coefficients are different, but the same sign, it is an ellipse. Complete the square to find the center.

$$9x^2 - 18x + 4y^2 + 16y = 11$$
$$9(x^2 - 2x) + 4(y^2 + 4y) = 11$$
$$9(x^2 - 2x + 1 - 1) + 4(y^2 + 4y + 4 - 4) = 11$$
$$9(x - 1)^2 - 9 + 4(y + 2)^2 - 16 = 11$$
$$9(x - 1)^2 + 4(y + 2)^2 = 36$$
$$\frac{(x - 1)^2}{2^2} + \frac{(y + 2)^2}{3^2} = 1 .$$

The center is (1, -2) (Figure 15). Because $3 > 2$ and 3 is associated with y, the major axis is vertical. $c^2 = 3^2 - 2^2 = 5$, so $c = \sqrt{5}$. The foci are $\sqrt{5}$ from the center.

The Equation of a Hyperbola. The "standard form" of a hyperbola centered at the origin with foci at $(-c, 0)$ and $(c, 0)$ follows by simplifying Equation 9.2.4. We omit the details, which are very similar to the details included for the ellipse (problem B17).

Hyperbolas Centered at the Origin with Foci on the x-axis. The "standard form" of a hyperbola centered at the origin and foci at $(c, 0)$ and $(-c, 0)$ is

(9.2.8)
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 .$$

where $a^2 + b^2 = c^2$. The lines $y = \pm(b/a)x$ are asymptotes (Figures 16 and 17).

This equation is very similar to the equation for an ellipse–except the plus sign has become a minus sign.

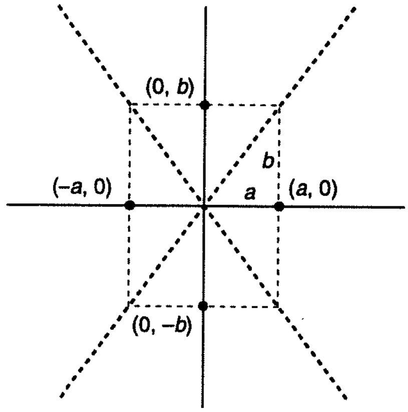


Figure 16: Sketching a hyperbola.
Step 1: Find $(a, 0)$ and $(-a, 0)$.
Step 2: Find $(0, b)$ and $(0, -b)$ and sketch in the rectangle, as shown.
Step 3: Lightly draw the asymptotes through the corners of the box and the origin.

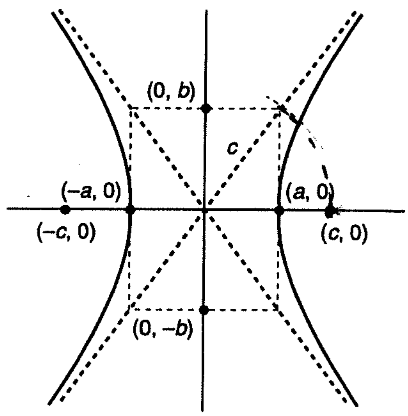


Figure 17: Sketching a hyperbola.
Step 4: Sketch the curve through $(a, 0)$ sweeping up and down toward the asymptotes.
Step 5: Fill in the other half, symmetrically.

The graph goes through two easy points: $(-a, 0)$ and $(a, 0)$. These points on the curve are between the foci (the reverse of the situation for an ellipse, where the foci are between the points on the curve). There is no y-value for $x = 0$ because substituting in $x = 0$ yields the equation $-y^2/b^2 = 1$, which cannot be satisfied (because a negative number is never 1). Therefore, the two branches of the hyperbola do not meet; there is a gap between them near $x = 0$.

Solving for y and simplifying

(9.2.9)
$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$$

This is the functional form suitable for graphing with a graphics calculator.

As x becomes large, $\sqrt{x^2 - a^2}$ approaches x, so, for large x, y is approximately $\pm(b/a)x$. The lines through the origin with equations $y = \pm(b/a)x$ are asymptotes with slopes $\pm b/a$. The curve sweeps up and down from the axis toward the asymptotes. To sketch the graph of Equation 9.2.8 it is essential to locate $(\pm a, 0)$ and the two asymptotes. An easy way to plot the asymptotes is to make the rectangle in Figure 16 and draw the diagonal line through the corners and the origin.

To locate the focus at $(c, 0)$, swing an arc centered at the origin through the upper right corner of the box and mark where the arc cuts the x-axis (Figure 17). This works because c is both the distance from the center to the foci and the distance from the center to the corner of the box (because $c^2 = a^2 + b^2$).

Calculator Exercise 2: Formula 9.2.9 gives the functional form which permits graphing a hyperbola with a graphics calculator. Identify a and b in the next example and use 9.2.9 to graph the equation on your calculator.

Example 7: Sketch the graph of the equation $x^2/25 - y^2/4 = 1$.
Here $a = 5$ and $b = 2$. The graph goes through (-5, 0) and (5, 0). The asymptotes have slope 2/5 and -2/5 (Figure 18).

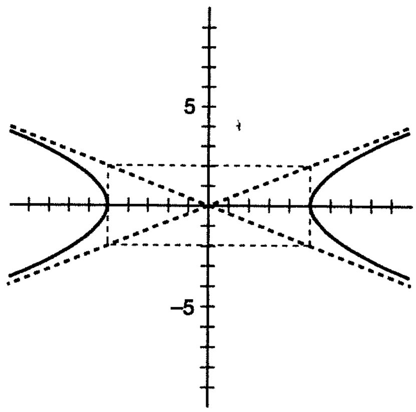


Figure 18: $x^2/25 - y^2/4 = 1$.
[-5, 5] by [-5, 5].

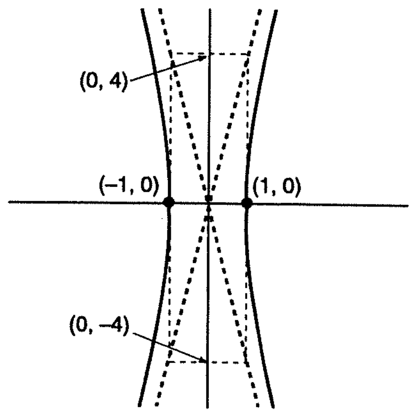


Figure 19: $x^2 - y^2/16 = 1$.
[-5, 5] by [-5, 5].

Example 8: Sketch the graph of the equation $x^2 - y^2/16 = 1$.
Here $a = 1$ and $b = 4$. The graph still opens around the x-axis because the "x^2" term is positive, as in Equation 9.2.8. The graph goes through (-1, 0) and (1, 0), and the asymptotes have slope 4 and -4 (Figure 19).

Equation 9.2.8 is *not* symmetric in x and y. One of the squared terms has a plus sign and the other a minus sign. The foci are on the axis corresponding to the plus sign. The hyperbola "opens" around the central axis, which is the horizontal axis in Figures 16 and 17. If the foci are vertically aligned, the roles of "x" and "y" are switched and the parabola opens around the vertical axis.

Example 9: Graph $y^2 - x^2 = 1$.
The graph opens around the y-axis (Figure 20).

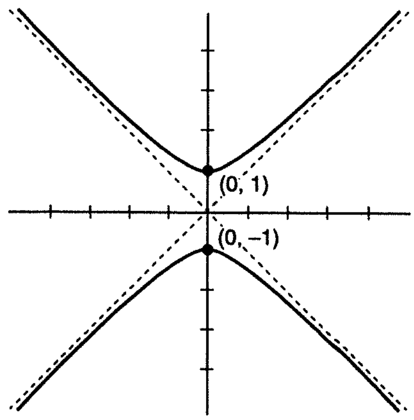


Figure 20: $y^2 - x^2 = 1$
opens around a vertical axis, corresponding to the positive y^2 term.
[-5, 5] by [-5, 5].

Scale Changes. In Section 2.2 (Composition and Decomposition) we discussed scale changes in general, and in Section 3.3 (Distance, Circles, and Ellipses) we discussed circles and how, with scale changes, circles become ellipses. The "standard forms" can be interpreted as scale changes of just two basic graphs, the unit circle "$x^2 + y^2 = 1$" (Figure 21) and the hyperbola "$x^2 - y^2 = 1$" (Figure 22).

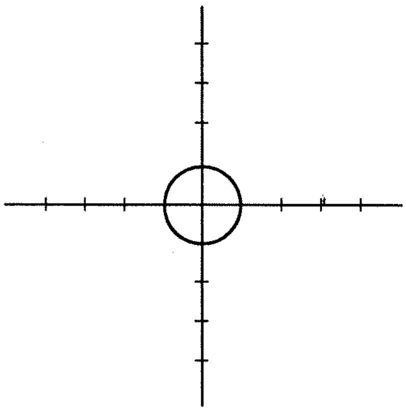


Figure 21: $x^2 + y^2 = 1$.
[-5, 5] by [-5, 5].

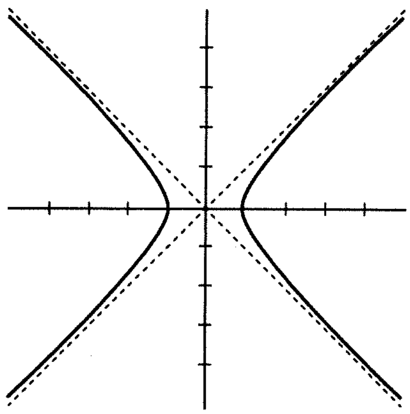


Figure 22: $x^2 - y^2 = 1$.
[-5, 5] by [-5, 5].

If, in an equation, "x" is replaced by "x/a," the new graph becomes a times as wide. This is because x must be a times as great for "x/a" to assume the same value and yield the same image as the old "x". Similarly, replacing "y" by "y/b" yields a graph b times as tall. Can you see this by comparing Figures 9 through 13 to Figure 21? Can you see this by comparing Figures 18 and 19 to Figure 22?

Location Changes. To shift the graph so the center is (h, k), simply substitute "$x - h$" for "x" and "$y - k$" for "k" in the form 9.2.8.

Hyperbola with a Horizontal Axis: The "standard form" of a hyperbola with a horizontal axis is

(9.2.10)
$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1\,.$$

The center is (h, k), the axis is $y = k$, the foci are $(h - c, k)$ and $(h + c, k)$, where $c^2 = a^2 + b^2$. The graph goes through $(h + a, k)$ and $(h - a, k)$. The asymptotes are lines through the center with slopes $\pm b/a$.

Example 10: Identify and sketch the hyperbola with equation
"$x^2 + 4x - 9y^2 = 32.$"

Aim for standard form. First complete the square.

$x^2 + 4x - 9y^2 = 32$ is equivalent to

$$x^2 + 4x + 4 - 9y^2 = 36,$$
$$(x + 2)^2 - 9y^2 = 36,$$
$$\frac{(x+2)^2}{6^2} - \frac{y^2}{2^2} = 1\,.$$

The center is (-2, 0). $a = 6$ and $b = 2$. The axis is horizontal, since the "x^2" term has the positive sign. Figure 23 illustrates the graph, including the asymptotes.

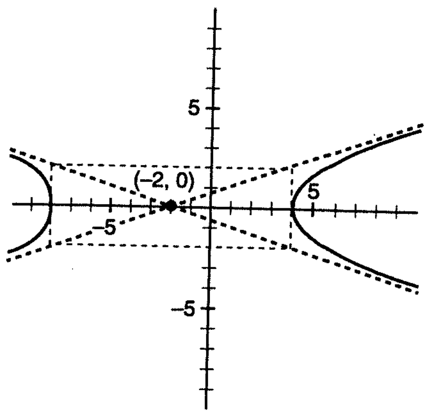


Figure 23: The graph of $x^2 + 4x - 9y^2 = 32$. [-10, 10] by [-10, 10].

Example 11: Identify and sketch the hyperbola with equation

$$\text{"}5x^2 - 2y^2 - 16y = 12.\text{"}$$

The orientation of this hyperbola is not easy to guess. Aim for standard form. Complete the square.

$$5x^2 - 2y^2 - 16y = 12$$
$$5x^2 - 2(y^2 - 8y) = 12.$$
$$5x^2 - 2(y^2 - 8y + 16 - 16) = 12.$$
$$5x^2 - 2(y^2 - 4)^2 = 12 - 32 = -20.$$

Note the negative sign on "-20." To convert to "standard form" the right side must be (positive) 1. The signs must be changed.

$$2(y - 4)^2 - 5x^2 = 20.$$

$$\frac{(y-4)^2}{10} - \frac{x^2}{4} = 1$$

$$\frac{(y-4)^2}{(\sqrt{10}\,)^2} - \frac{x^2}{2^2} = 1\,.$$

The center is (0, 4) and it opens around a vertical axis (Figure 24).

The equation we just obtained in Example 11 is very similar to the standard form for a horizontal hyperbola–but the signs are changed on the squared terms. That switches the roles of the axes and makes the hyperbola vertical.

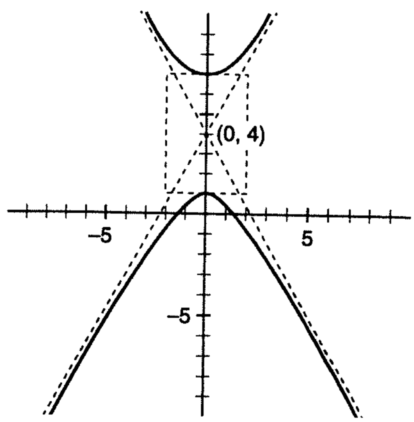


Figure 24: $5x^2 - 2y^2 - 16y = 12$. [-10, 10] by [-10, 10].

Hyperbola with a Vertical Axis: The "standard form" of a hyperbola with a vertical axis is

$$\frac{(y - k)^2}{b^2} - \frac{(x - h)^2}{a^2} = 1 . \qquad (9.2.11)$$

The center is (h, k), the axis is $x = h$, the foci are $(h, k + c)$ and $(h, k - c)$, where $c^2 = a^2 + b^2$. The graph goes through $(h, k + c)$ and $(h, k - c)$. The asymptotes are lines through the center with slopes $\pm b/a$.

Hyperbolas and ellipses have important reflective properties which are discussed next.

Reflective Properties. All ellipses have a remarkable reflective property. Every ray generated at one focus will reflect through the other focus, regardless of its original direction (Figure 25).

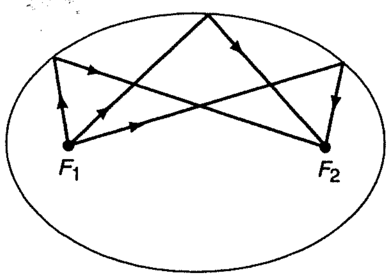


Figure 25: Rays generated at one focus of an ellipse reflect through the other focus.

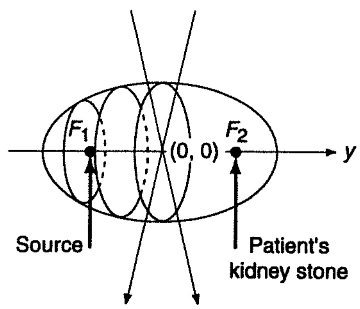


Figure 26: An ellipsoidal lithotripter. Shock waves reflect through the second focus at the same time.

Example 12: A lithotripter is a medical machine for destroying painful kidney stones without surgery. It has a reflecting ellipsoidal surface (Figure 26). An ellipsoid is a 3 dimensional surface formed by rotating an ellipse about its major axis. The reflecting surface is not a complete ellipsoid. It encloses only one focus and leaves room for the other focus to be precisely positioned at the kidney stone inside the patient. Then a strong shock wave is generated at the enclosed focus. Waves reflecting off the ellipsoid in all directions pass through the other focus (the kidney stone) at the same time. The focused shock wave can pulverize the stone and then its fragments can pass through the system without surgery.

Hyperbolas also have a remarkable and useful reflective property. Consider one branch with the convex side reflective. All light rays aimed at the focus behind the branch will be reflected through the other focus, regardless of the original angle of the rays (Figure 27).

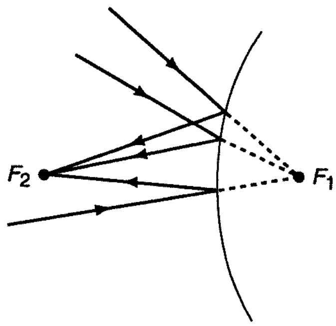


Figure 27: Rays aimed at a focus of a hyperbola reflect through the other focus.

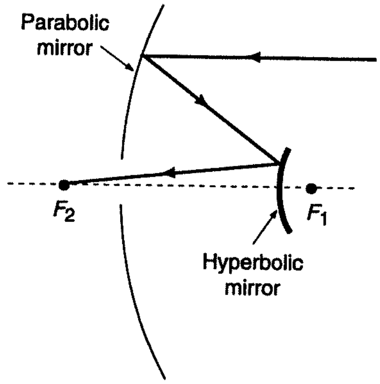


Figure 28: Parabolic and hyperbolic mirrors combined.

Example 13: A reflecting telescope with a parabolic mirror at one end will focus parallel rays at a point in front of the mirror (Figure 9.1.15). But you can't put your eye there; your head would block the incoming rays. Instead, put a convex hyperbolic mirror just in front of the focus so that the focus of the parabola is also the focus of the hyperbola (Figure 24). Then the rays reflected from the parabolic mirror will be aimed at a focus of the hyperbolic mirror. By the reflective property of hyperbolas, they will reflect through the other focus. Arrange it so that the second focus of the hyperbola is behind the parabolic mirror. Then, only a small hole in the big parabolic mirror is needed to focus the light behind the big mirror, where an eye or camera can be located.

Distinguishing the Types of Conic Sections. The algebraic equations for parabolas, circles, ellipses, and hyperbolas are somewhat similar. How can we tell which is which?

In the equation of a parabola, only one of "x" or "y" is squared (see 9.1.5 and 9.1.6). If both "x" and "y" are squared, expect a circle, ellipse, or hyperbola. (There are some special cases where the graph is a line, a point, or nothing at all.) How are circles, ellipses, and hyperbolas distinguished?

Consider an equation of the form

(9.2.12) $$ax^2 + bx + cy^2 + dy + k = 0.$$

Assuming $a \neq 0$ and $c \neq 0$ (that is, both squared terms exist)

1) If a and c are the same sign, expect an ellipse (problem B16).
2) If a and c are equal, expect a circle, which is a special ellipse.
3) If a and c are opposite signs, expect a hyperbola.

One of the squared terms may be missing.

4) If $a = 0$ or $c = 0$, but not both, expect a parabola.

To determine the equation in "standard form," the procedure is to first complete the square on x and also on y to locate the center. Then aim for one of the standard forms.

Example 14: $3x^2 + 12x + 3y^2 = 50$ is a circle. The coefficients on x^2 and y^2 are equal.

$5x^2 + 20x + y^2 = 50$ is an ellipse because the coefficients on x^2 and y^2 are not equal, but they have the same sign.

$5(x + 2)^2 - y^2 = 50$ is a hyperbola. The coefficients on x^2 and y^2 have opposite signs. Because the coefficient on x^2 is the same sign as the constant, the hyperbola opens around the x-axis.

$x + 2 + y^2 = 50$ is a parabola. There is no x^2 term.

Conclusion. Equations with terms in both x^2 and y^2 can determine circles, ellipses, or hyperbolas. These shapes have important properties. They are easily graphed after they are written in "standard form." Completing the square is an important step in attaining standard form.

Terms: ellipse, hyperbola, focus, semi-major axis, semi-minor axis, standard form.

Exercises for Section 9.2, "Ellipses and Hyperbolas":

^^^^Determine the type of conic section. Rewrite the equation in "standard form," determine a, b, and c, identify the center, and roughly sketch it.

A1. $x^2 + y^2/4 = 1$. [$c = \sqrt{3}$]
A2. $x^2/9 + y^2 = 1$. [$a = 3$]
A3. $x^2/4 - y^2 = 1$. [$a = 2$]
A4. $x^2 - y^2/9 = 1$. [$b = 3$]
A5. $y^2 - x^2/4 = 1$. [$c = \sqrt{5}$]
A6. $y^2/16 - x^2 = 1$. [$a = 1$]
A7. $4x^2 + 9y^2 = 36$. [$b = 2$]
A8. $4x^2 + y^2 = 25$. [$c = 4.3$]
A9. $4x^2 - 9y^2 = 36$. [$c = 3.6$]
A10. $4x^2 - y^2 = 25$. [$c = 5.6$]
A11. $x^2 - 4y^2 = 1$. [$c = 1.1$]
A12. $9x^2 - y^2 = 1$. [$c = .94$]
A13. $100x^2 + 25y^2 = 4$. [$c = .35$]
A14. $4x^2 - 9y^2 = 1$. [$b = 1/3$]
A15. $(x - 3)^2 + 4(y + 2)^2 = 4$. [$c = 1.7$]
A16. $9(x + 2)^2 - y^2 = 9$. [$c = 3.2$]
A17. $4x^2 - 8x + 9y^2 + 36y + 4 = 0$. [$c = 2.2$]
A18. $4x^2 + 8x + y^2 - 6y + 9 = 0$. [$c = 2.2$]

^^^^Find the equation of the ellipse with the following properties.

A19. Foci (-2, 0) and (2, 0), semi-major axis 3.
A20. Foci (0, 1) and (0, -1), semi-major axis 2.
A21. Center (1, 3), semi-major axis vertical and length 4, semi-minor axis 2.
A22. Center (3, 4), semi-major axis horizontal and length 5, semi-minor axis 1.

^^^^Find the equation of the hyperbola with the following properties.
A23. Foci at (-2, 0) and (2, 0), through (1, 0).
A24. Foci at (0, 1) and (0, -1), through (0, ½).
A25. Center at (1, 5), through (1, 9), with focus at (1, 10).
A26. Center at (-2, 3), through (0, 3), with focus at (1, 3).

^^^^Identify the type of conic section. Do as little work as possible.
A27. $x^2 - 5y^2 = 1$.
A28. $x^2 + 5y = 2$.
A29. $x^2 + y^2 + 10y = 50$.
A30. $x^2 + 6x + 3y^2 = 100$.
A31. $x + 5y^2 = 27$.
A32. $y^2 + 6y = x^2 + 10x$.
A33. $3x^2 + 9x + 3y^2 - 8y = 200$.
A34. $3x^2 + 9x + 4y^2 - 8y = 200$.
A35. $3x^2 + 9x - 4y^2 - 8y = 200$.
A36. $3x^2 + 9x - 8y = 200$.

^^^^^^^^

B1.* a) Sketch an ellipse centered at the origin and label a, b, and c on it. Include a right triangle that relates a, b, and c as its sides and label the hypotenuse.

B2.* a) Define *ellipse*. b) Illustrate the definition with a sketch.

B3.* a) Sketch a hyperbola centered at the origin and label a, b, and c on it. Include a right triangle that relates a, b, and c as its sides and label the hypotenuse.

B4.* a) Define *hyperbola*. b) Illustrate the definition with a sketch.

B5. Suppose $x^2/c^2 - y^2/d^2 = 1$. What are the slopes of the asymptotes?

B6. Suppose $y^2/c^2 - x^2/d^2 = 1$. What are the slopes of the asymptotes?

B7. Let F be (3, 0) and L be the y-axis. Let P be a point such that the distance from F to P is 1/2 the distance from P to L. a) Find, by inspection, two points on the x-axis that satisfy the restriction on P. b) Find, by inspection, two points on the line $x = 3$ that satisfy the restriction on P. c) Use rectangular coordinates to set up an equation for all such points. d) Simplify the equation into standard form and identify the conic section.

B8. Let F be (3, 0) and L be the y-axis. Let P be a point such that the distance from F to P is twice the distance from P to L. a) Find, by inspection, two points on the x-axis that satisfies the restriction on P. b) Find, by inspection, two points on line $x = 3$ that satisfy the restriction on P. c) Use rectangular coordinates to set up an equation for all such points. d) Simplify the equation into standard form and identify the conic section.

^^^^[For B9-B15] The <u>eccentricity</u> of an ellipse or hyperbola is

$$e = \frac{\textit{the distance between foci}}{\textit{the distance between vertices}},$$

where the vertices are the points on the major axis.
B9. For an ellipse, if $a > b$, $e = c/a$. Suppose the eccentricity of an ellipse is 1/2 and $a > b$. a) What is the ratio of a to b? b) If $a/b = 2$, what is the eccentricity?

B10. [See the definition of "eccentricity" above B9.] For a hyperbola that fits form 9.2.10, if the eccentricity is 3, what is the ratio of a to b? b) If $a/b = 3$, what is the eccentricity?

B11. For a hyperbola with horizontal axis, give a formula for the slopes of the asymptotes if the eccentricity is $e > 1$.

B12. For ellipses, give a formula for the ratio of length to width if the eccentricity is e.

B13. If the ratio of length to width of an ellipse is r, give a formula for the eccentricity.

B14. If the slopes of the asymptotes of a horizontal hyperbola are $\pm m$, give a formula for its eccentricity.

B15. Which hyperbola has greater eccentricity, the one in Figure 18 or the one in Figure 19?

B16. In 9.2.12 "special cases" can occur that do not produce the "expected" result.
a) Case 1 may produce a point or nothing. How?
b) Case 3 may produce a pair of lines. How?

B17. Inspect the details required to convert 9.2.3 into standard form (page 511). To convert 9.2.4 to standard form most steps would be very similar. At which step is the really significant difference? Redo that step to show you understand the difference.

B18. Let F be the origin and L be the line $x = -h$. Let P be a point such that the distance from F to P is e (a parameter, the "eccentricity," greater than 0) times the distance from P to L. Use rectangular coordinates to set up an equation for all such points. [Do not bother to simplify. Simplifying is quite hard.]

B19. In Example 1 the time of the burst was recorded at F_1 and compared to the time its reflection was received at F_2 to determine the total distance the sound traveled, and therefore, to determine the ellipse (Figure 3). There is a way to determine the total distance by recording time only at F_2, since the sound will be heard twice at F_2, one directly and once reflected. a) How can we do this, even if we fail to record when we generated the burst at F_1? b) For example, if sound travels at 1100 feet per second, F_1 is 2200 feet from F_2, and we record the sound twice at F_2, the second time 1/2 second after the first, what do we know that allows us to find the equation for all possible reflecting points?

B20. The graph of $y = 1/x$ is a hyperbola. a) What is its axis of symmetry?
b) Let $u = x - y$ and $v = x + y$. Rewrite "$y = 1/x$" in terms of u and v. c) Graph $u = 0$ and $v = 0$, and $y = 1/x$ together. [Note how, on the axis system created in Part (c), the graph looks like a standard hyperbola opening up around an axis.]

B21. [See Figure 4] If F_1 is the both center of a circle and the focus of an (non-circular) ellipse, can the circle and ellipse intersect in four locations, as some circles and ellipses can? Argue why it can or can not.

B22. (Short and interesting, but hard) A hyperbola can be determined as a cross-section of a cone (Figure 9.1.2C). Explain, using this fact, why the tip of the shadow of a pointed object moves in a branch of a hyperbola as the sun goes across the sky. For example, the tip of the shadow of the point sticking up from a flat sundial moves in a hyperbolic path.

Section 9.3. Polar Equations of Conic Sections

Parabolas, ellipses, and hyperbolas can be described using distances from a focus and a directrix. When the focus is located at the origin, polar coordinates use r for the distance from the focus. This makes conic sections easy to describe in polar coordinates.

There are several equivalent ways to define conic sections. Here is one that uses a focus and a directrix to simultaneously describe parabolas, ellipses, and hyperbolas.

Let L be a line (a directrix) and F be a point (a focus) not on that line. Let e, a parameter that is called the eccentricity of the conic section, be an arbitrary constant greater than 0. Consider the set of points P such that the distance from F to P is e times the distance from P to L.

(9.3.1) $$\frac{\textit{distance from } F \textit{ to } P}{\textit{distance from } P \textit{ to } L} = e$$

If $e = 1$, the curve is a parabola. If $e < 1$, the curve is an ellipse. If $e > 1$, the curve is a hyperbola.

If $e = 1$, we already know that $FP = PL$ defines a parabola (Figure 1, Definition 9.1.3). However, this way to define ellipses and hyperbolas is new. It yields simple polar-coordinate equations.

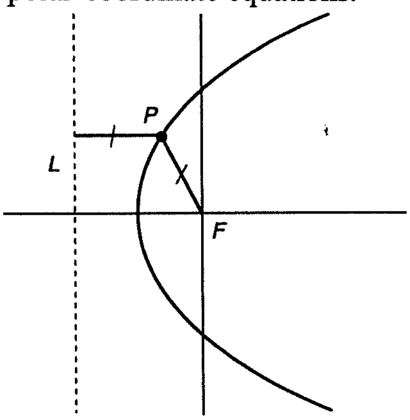


Figure 1: A parabola. $e = 1$.

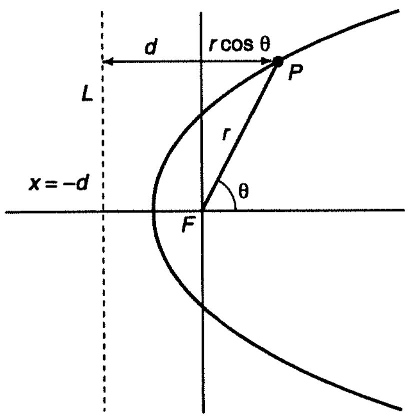


Figure 2: The distances from F to P and from P to L.

Orient the axis system so that F is at the origin and the directrix is $x = -d$ (Figure 2). Consider P to the right of the directrix as in Figure 2. Now express this: "The distance from F to P is e times the distance from P to L."

In polar coordinates, $P = (r, \theta)$ and the distance from F to P is simply r. The distance from P to L is $r \cos \theta + d$ (because $x = r \cos \theta$, Figure 2). Therefore,

$$r = e(r \cos \theta + d).$$

Solving for r in terms of θ,

$$r - er \cos \theta = ed.$$
$$r(1 - e \cos \theta) = ed.$$

(9.3.2) $$r = \frac{ed}{1 - e \cos \theta}$$

This is the <u>polar form</u> of the equation of a conic section with focus (0, 0), directrix $x = -d$, and eccentricity $e > 0$. If $e < 1$, it is an ellipse. If $e = 1$, it is a parabola. If $e > 1$, it is a hyperbola.

Example 1: Let a focus be at the origin, the directrix be $x = -3$, and the eccentricity be 1/2. Find the polar equation of the conic section and sketch it.

The polar-coordinate equation follows directly from 9.3.2 (Figure 3):

$$r = \frac{(1/2)3}{1 - (1/2)\cos \theta}$$

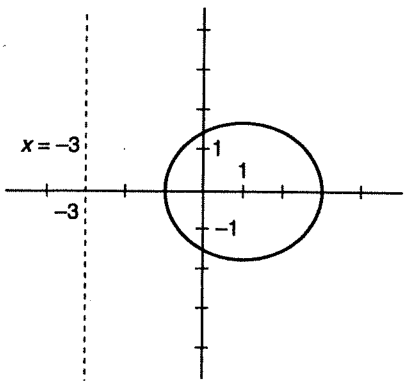


Figure 3: $r = (3/2)/[1 - (1/2)\cos \theta]$. $[-5,5]$ by $[-5,5]$. $0 \le \theta < 2\pi$. $e = 1/2$.

Four points are easy to find because cosine is so simple at the major angles 0, $\pi/2$, π, and $3\pi/2$. When $\theta = 0$, $\cos \theta = 1$ and the polar-coordinate point (3, 0) is determined (it is also represented by (3, 0) in rectangular coordinates). When $\theta = \pi/2$, $\cos \theta = 0$, and the polar-coordinate point $(3/2, \pi/2)$ results, which is on the positive y-axis. When $\theta = \pi$, $\cos \theta = -1$ and the point $(1, \pi)$ results (which is (-1, 0) in rectangular coordinates). Finally, $\theta = 3\pi/2$, yields the point $(3/2, 3\pi/2)$ on the negative y-axis (which is (0, -3/2) in rectangular coordinates).

Note how these four points fit the fact that the eccentricity is 1/2. In rectangular coordinates, the point (3, 0) is 3 units from the origin and 6 units from the line $x = -3$, for a ratio of 1/2. On the negative x-axis, (-1, 0) is 1 unit from the origin and 2 units from the directrix, yielding the desired ratio 1/2. Points on the y-axis are all 3 units from the directrix. Therefore, to make the ratio 1/2, the points on the-axis must be 3/2 units from the origin: (0, 3/2) and (0, -3/2) in rectangular coordinates.

Example 2: Identify the type of conic section and give its eccentricity and directrix (Figure 4).

$$r = \frac{12}{3 - 2\cos\theta}.$$

Manipulate this into the form of 9.3.2. First get the "1" in the denominator.

$$r = \frac{12}{3 - 2\cos\theta}$$

$$r = \frac{4}{1 - (2/3)\cos\theta}.$$

So $e = 2/3 < 1$, so it is an ellipse. Now, since the numerator is ed, $(2/3)d = 4$, so $d = 6$. The directrix is $x = -6$.

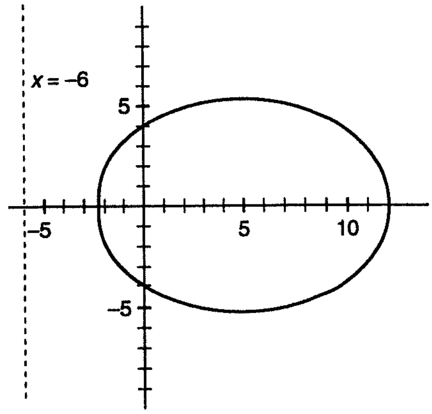


Figure 4:
$r = 12/(3 - 2\cos\theta)$.
[-7,13] by [-10,10].
$0 \le \theta < 2\pi$. $e = 2/3$.

Example 3: Identify the type of conic section and give its eccentricity and directrix (Figure 5).

$$r = \frac{5}{2 - 4\cos\theta}.$$

Again, get the "1" in the denominator.

$$r = \frac{5/2}{1 - 2\cos\theta}.$$

So $e = 2 > 1$ and the conic section is a hyperbola. $5/2 = ed = 2d$, so $d = 5/4$. The directrix is $x = -5/4$.

Discovery 1: Graph the equation in Example 3 (Figure 5) with a graphics calculator in "Polar" mode. Which quarter of the hyperbola appears first? Second? Explain why, mathematically (problem B1). [If your calculator does not have a "Polar" mode, see problem B31.]

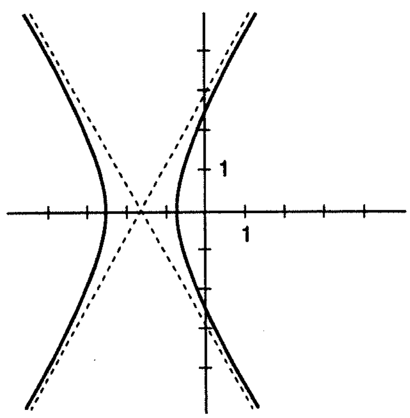


Figure 5: $r = 5/(2 - 4\cos\theta)$.
[-5,5] by [-5,5]. $0 \le \theta < 2\pi$.
$e = 2$.

Figures 6 and 7 illustrate the effect of eccentricity on the shapes of conic sections.

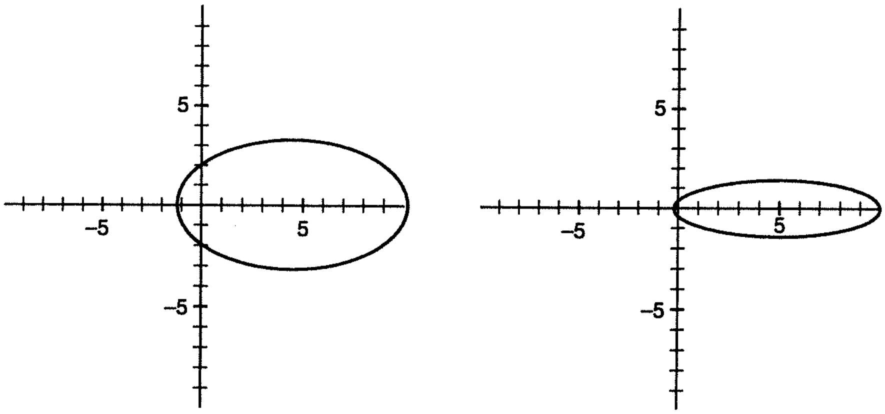


Figure 6: Ellipses with $e = .8$ and $.95$, respectively.

$$r = \frac{2}{1 - .8 \cos \theta}. \qquad r = \frac{.5}{1 - .95 \cos \theta}.$$

[-10, 10] by [-10, 10]. $0 \le \theta < 2\pi$.

Discovery 2: Figure 6 displays the shapes associated with 2 different values of e. To get both ellipses approximately the same size in the same window, the numerators had to be different. Why? Use a graphics calculator to compare the graphs of

$$r = \frac{2}{1 - .8 \cos \theta} \text{ and } r = \frac{2}{1 - .95 \cos \theta}.$$

Why is the second so much wider than the first (problem B3)?

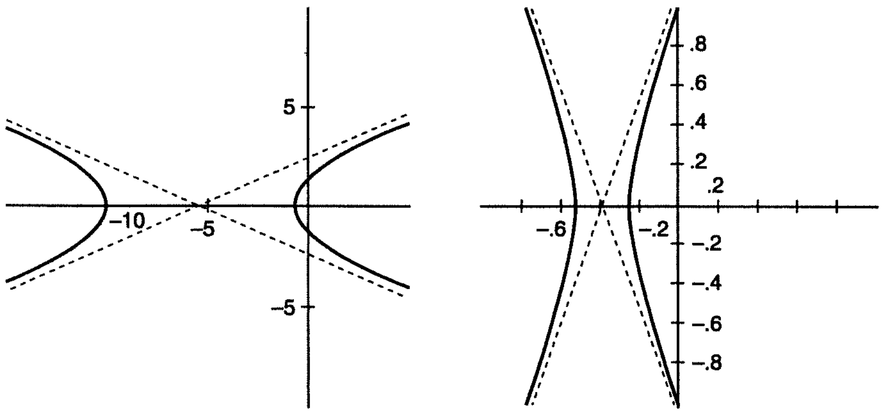


Figure 7: Hyperbolas with $e = 1.1$ and 3, respectively.

$$r = \frac{1}{1 - 1.1 \cos \theta}. \qquad r = \frac{1}{1 - 3 \cos \theta}.$$

[-15, 5] by [-10, 10]. $0 \le \theta < 2\pi$. [-1, 1] by [-1, 1].

Other Orientations. The family of conic sections described by 9.3.2 had a vertical directrix $d > 0$ units to the *left* of the origin. Similar equations can be derived if the directrix is vertical and to the right of the origin, or if it is horizontal and above or below the origin.

Let the eccentricity be $e > 0$ and the directrix be $d > 0$ units from the focus at the origin. The usual equation for an ellipse is given by 9.3.2. However, if the directrix is vertical at $x = d$ (instead of $x = -d$, as 9.3.2) an equation is

$$r = \frac{ed}{1 + e \cos \theta} . \tag{9.3.3}$$

If the directrix is horizontal at $y = -d$, an equation is

$$r = \frac{ed}{1 - e \sin \theta} . \tag{9.3.4}$$

Finally, if the directrix is horizontal at $y = d$, an equation is

$$r = \frac{ed}{1 + e \sin \theta} . \tag{9.3.5}$$

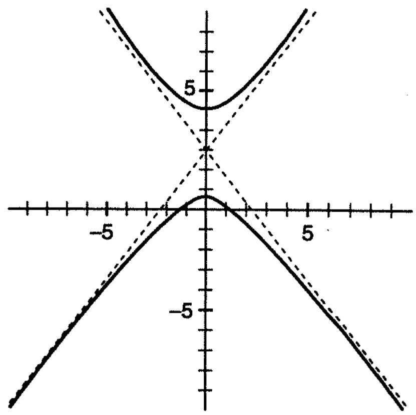


Figure 8: $r = 5/(4+5\sin \theta)$.
[-10, 10] by [-10, 10].
$0 \le \theta < 2\pi$. $e = 5/4 = 1.2$.

These equations are given for positive values of d, but they also work for negative values of d (problems B7, B8, and B24).

A vertical directrix leads to symmetry about the x-axis. In this case r is written as a function of cosine, which is symmetric about $\theta = 0$, which is the x-axis. A horizontal directrix leads to symmetry about the y-axis. In that case, r is written as a function of sine, which is symmetric about $\theta = \pi/2$, which is the y-axis.

Example 4: Find the eccentricity and directrix of conic section with equation

$$r = \frac{5}{4 + 5\sin\theta} .$$

Aim for the "1" in the denominator.

$$r = \frac{5/4}{1 + (5/4)\sin\theta}$$

$$r = \frac{(5/4)1}{1 + (5/4)\sin\theta} .$$

Therefore, from 9.3.5, $e = 5/4 > 1$, the conic section is a hyperbola, and the directrix is $y = 1$ (Figure 8).

The Major Axis. The semi-major axis played an important part in rectangular-coordinate equations of conics. Here we discover the length, a, of the semi-major axis in terms of e and d from equation 9.3.2.

Example 5: Find the length of the semi-major axis when the polar equation is

$$r = \frac{4}{1 - (1/3)\cos\theta}.$$

This equation fits 9.3.2, so the directrix is vertical and the major axis of length $2a$ is horizontal (9.2.7, Figure 9). Therefore $2a$ is the distance between the points on the x-axis, that is, the distance from the point where $\theta = 0$ to where $\theta = \pi$. Add the corresponding values of r:

$$2a = r(0) + r(\pi)$$

$$= \frac{4}{1 - (1/3)\cos 0} + \frac{4}{1 + (1/3)\cos\pi}$$

$$= \frac{4}{2/3} + \frac{4}{4/3}$$

$$= 6 + 3 = 9.$$

Therefore $a = 9/2$.

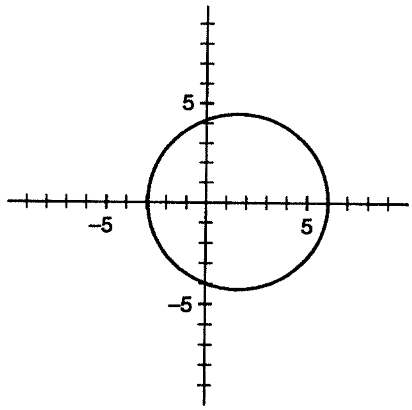


Figure 9:
$r = 4/[1 - (1/3)\cos\theta]$.
$[-10, 10]$ by $[-10, 10]$.
$0 \le \theta < 2\pi$. $e = 1/3$.

For general ellipses in the format 9.3.2,

$$2a = r(0) + r(\pi) = \frac{ed}{1 - e\cos 0} + \frac{ed}{1 - e\cos\pi}$$

$$= \frac{ed}{1 - e} + \frac{ed}{1 + e}$$

$$= ed\,\left(\frac{1 + e + 1 - e}{1 - e^2}\right)$$

$$= \frac{2ed}{1 - e^2}.$$

This generalizes to ellipses and hyperbolas in the forms 9.3.2-9.3.5 (Problem B4). If the directrix is $|d|$ units from the origin and the eccentricity is e, then

$$a = \frac{e|d|}{|1 - e^2|} \tag{9.3.6}$$

where a is the length of the semi-major axis. Also,

$$e|d| = a|1 - e^2|. \tag{9.3.7}$$

Example 6: Pluto moves about the sun in an ellipse with eccentricity 0.2485 and a = 39.5 astronomical units. (The distance from the Earth to the Sun defines 1 astronomical unit.) Give a polar equation of the orbit (Figure 10).

The parameters in polar form are e and d, and d only appears in the product ed. Use 9.3.7.

$$r = \frac{39.5(1 - .2485^2)}{1 - .2485\cos\theta}$$

$$= \frac{37.06}{1 - .2485\cos\theta}.$$

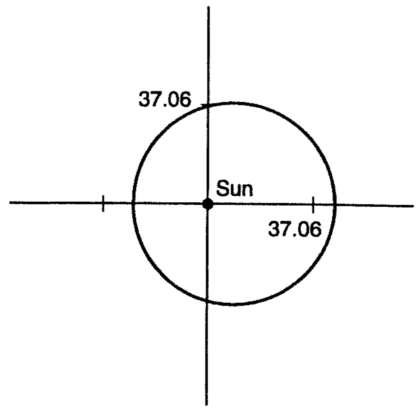


Figure 10: The orbit of Pluto. $e = .2485$.

The rectangular-coordinate equations for ellipses and hyperbolas in Section 9.2 emphasize the center, in contrast to the polar equations in this section which emphasize a focus. So the polar equations in this section are not simply polar versions of the rectangular-coordinate equations; the location and parameters are different.

Example 5, revisited: Find the equation in standard rectangular-coordinate form if, as in Example 5 and Figure 9, $r = \dfrac{4}{1 - (1/3)\cos\theta}$.

To find rectangular-coordinate form we determine a, b, and the center (h, k) of form 9.2.7.

In Example 5 we found $a = 9/2$. Formula 9.3.6 would yield the same result in one step: $4/(1 - (1/3)^2) = 9/2$. The plan is to find the center next, which yields c, and then to find b from a and c.

The center is half way between the two endpoints of the major axis (Figure 9). The left endpoint has $x = -3$ and the right endpoint has $x = 6$, so the center has $x = 3/2$; it is (3/2, 0) (which is expressed the same way in both coordinate systems).

This tells us that $c = 3/2$, because "c" denotes the distance from the center to a focus. Now we can find b using a and c: $c^2 = a^2 - b^2$ (from 9.2.7). Plugging in,

$$(3/2)^2 = (9/2)^2 - b^2.$$
$$b^2 = (9/2)^2 - (3/2)^2.$$
$$b = \sqrt{18} = 4.24.$$

Therefore, the rectangular-coordinate equation is

$$\frac{(x - 3/2)^2}{(9/2)^2} + \frac{y^2}{(\sqrt{18})^2} = 1 .$$

Calculator Exercise 1: Solve for y and graph this on your calculator using rectangular coordinates. Compare your graph to Figure 9.

Eccentricity in Rectangular Coordinates. Eccentricity is a parameter of standard polar form but not of standard rectangular-coordinate form. In Section 9.2 where rectangular form was used, ellipses and hyperbolas were defined with two foci and the *center* was at the origin. In this section, they are defined using a directrix and one focus, where the *focus* is at the origin. Of course, it is possible to relate the parameters with which the two forms are expressed (problem B32). For example, the eccentricity e of polar form has a simple rectangular-coordinate expression using the notation of Section 9.2.

$$e = \frac{\textit{distance between foci}}{\textit{length of major axis}} .$$

(9.3.8) If the major axis is horizontal, $e = c/a$.
If the major axis is vertical, $e = c/b$.

Recall that c is the distance from the center to a focus, and the length of the semi-major axis is a or b (9.2.5 and 9.2.7).

Example 7: Find the rectangular-coordinate equation of a hyperbola centered at the origin, with horizontal major axis, $a = 4$ and $e = 1.25$.

The parameters of rectangular-coordinates form are a and b (9.2.5). We can use the given a and e to find c from 9.3.8, and then a and c to get b from 9.2.5 (Figure 11).

$$e = c/a \quad \text{[from 9.3.8]}$$
$$1.25 = c/4.$$
$$c = 5.$$
$$a^2 + b^2 = c^2 \quad \text{[from 9.2.7]}$$
$$4^2 + b^2 = 5^2.$$
$$b = 3.$$

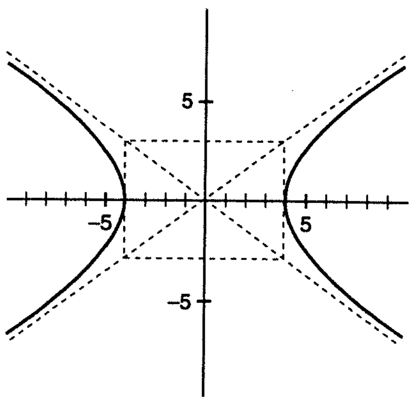


Figure 11: $x^2/4^2 - y^2/3^2 = 1$.
[-10,10] by [-10,10].
$e = 1.25$.

The equation is, from 9.2.7,

$$\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1 .$$

Example 8: Find the rectangular-coordinate equation of an ellipse with $e = 0.7$, one focus at the origin, the other focus on the positive x-axis, and semi-major axis 4 (Figure 12).

The center is c units from the focus, so the center is at $(c, 0)$ where $e = c/a$ from 9.3.8. Solving "$0.7 = c/4$," we find $c = 2.8$.

Also, $b^2 = a^2 - c^2$ for a horizontal ellipse, so $b^2 = 4^2 - 2.8^2 = 8.16$, and $b = \sqrt{8.16} = 2.857$. Plugging in to standard rectangular form (9.2.6):

$$\frac{(x - 2.8)^2}{4^2} + \frac{y^2}{2.857^2} = 1 .$$

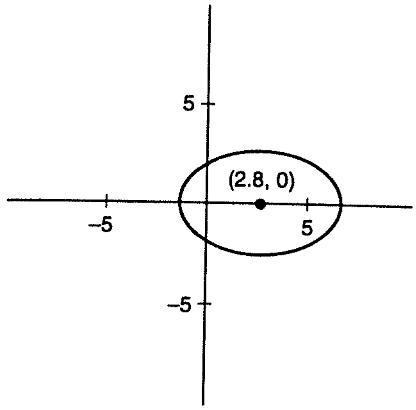


Figure 12: $e = .7$, $a = 4$, and one focus is at the origin. [-10, 10] by [-10, 10].

__**Circles**__. Ellipses with very small eccentricities are almost circles. Circles have eccentricity zero. But putting zero in for e in our polar forms yields "$r = 0$", an equation that describes the origin which is a single point, not a circle. So, how does the polar equation of a circle compare to the equation of an ellipse (9.3.2) with very small eccentricity?

The polar equation of a circle centered at the origin is "$r = k$," where k is the radius. θ does not appear. Reconsider equation 9.3.2:

$$r = \frac{ed}{1 - e \cos \theta}$$

To represent "$r = k$" the "$\cos \theta$" term would have to disappear, so e would have to be zero. But "ed" would have to be the radius k, which cannot hold if $e = 0$. So 9.3.2 does not hold for circles. But, we can imagine how, with smaller and smaller e, the effect of the "$e \cos \theta$" would become negligible. But the "ed" term on the top would also disappear unless we simultaneously adjust d (the location of the directrix) so that ed remains constant. So, to get ellipses that are similar in size to a given circle with fixed radius k, we need to increase d as we decrease e so that ed approaches k, the radius we want. That forces d to go to "infinity" as e goes down to zero, so ellipses that closely resemble circles have directrixes far from the origin (problems B21-23).

__**Conclusion**__. Polar coordinates make conic sections easy to describe if they have a focus at the origin. The eccentricity, e, is an important parameter.

Terms: parabola, ellipse, hyperbola, eccentricity, focus, directrix.

Exercises for Section 9.3, "Polar Equations of Conic Sections":

^^^^Identify the type of conic section, its eccentricity, and sketch its graph.

A1. $r = 1/(1 - \cos\theta)$.
A2. $r = 4/(2 - 2\cos\theta)$.
A3. $r = 1/(1 - 2\cos\theta)$.
A4. $r = 6/(3 - 2\cos\theta)$.
A5. $r = 1/(1 - .5\cos\theta)$.
A6. $r = 5/(3 - 4\cos\theta)$.

^^^^Identify the type of conic section, its eccentricity, and its directrix.

A7. $r = 5/(1 + \cos\theta)$. [dir: $x = 5$]
A8. $r = 1/(5 + 5\sin\theta)$. [dir: $y = 1/5$]
A9. $r = 7/(3 - 2\sin\theta)$. [$e = 2/3$]
A10. $r = 50/(7 + 2\sin\theta)$. [dir: $y = -25$]
A11. $r = 11/(9 + 10\sin\theta)$. [$e = 10/9$]
A12. $r = 3/(2 + 3\cos\theta)$. [dir: $x = 1$]

^^^^Find a polar equation of the conic satisfying the given conditions. All have a focus at the origin (pole).

A13. Parabola, directrix $y = -2$.
A14. Parabola, directrix $x = -10$.
A15. $e = .95$, directrix $x = 3$.
A16. $e = .1$, directrix $y = -2$.
A17. Directrix $y = 4$, $e = 1.5$.
A18. Directrix $x = 10$, $e = 1.01$.

Table of eccentricities and distances (in astronomical units) of the planets from the sun.

Mercury $e = .206$, $a = .387$.	Venus $e = .0068$, $a = .723$.
Earth $e = .0167$, $a = 1$.	Mars $e = .093$, $a = 1.52$.
Jupiter $e = .048$, $a = 5.20$.	Saturn $e = .056$. $a = 9.54$.
Uranus $e = .047$, $a = 19.2$.	Neptune $e = .0085$, $a = 30.07$.
Pluto $e = .2485$, $a = 39.5$.	

Find a polar equation of the orbit of

A19. Mercury
A20. Mars
A21. Venus.
A22. Jupiter

A23. Find the polar-coordinate equation of the ellipse in Example 8.

^^^^^^^^

B1. Do Discovery 1: Graph the equation in Example 3 (Figure 5) with a graphics calculator in polar mode. Which quarter of the hyperbola appears first? Second? Explain why, mathematically.

B2. Find the eccentricity of the famous hyperbola: $x^2 - y^2 = 1$.

B3. Do Discovery 2: Figure 6 displays the shapes associated with 2 different values of e. To get both pictures on about the same scale, the numerators had to be different. Why? Use a graphics calculator to compare the graphs of

$$r = \frac{2}{1 - .8\cos\theta} \quad \text{and} \quad r = \frac{2}{1 - .95\cos\theta}.$$

Why, mathematically, is the second so much wider than the first?

B4. For an ellipse in the form 9.3.2, $2a = r(0) + r(\pi)$, as in Example 5. a) Why isn't it $r(0) - r(\pi)$? b) Express $2a$ in terms of r for a hyperbola in the form 9.3.2.

B5.* [About 9.3.2 and 9.3.3] "$r = f(\cos\theta)$" denotes a composite function, for any function f. Because of the "$\cos\theta$" part, it always has a certain symmetry. a) What symmetry does the graph have? b) Why?

B6.* [About 9.3.4 and 9.3.3] "$r = f(\sin\theta)$" denotes a composite function, for any function f. Because of the "$\sin\theta$" part, it always has a certain symmetry. a) What symmetry does the graph have? b) Why?

B7. [Compare 9.3.2 and 9.3.3] a) Give the equation if the directrix is $x = 3$ and $e = 2$, using 9.3.3. b) Graph it. c) Suppose we treat $x = 3$ as $x = -(-3)$ and use 9.3.2. Give the equation. d) Graph it. e) How do the two graphs differ? f) Explain the answer to part (e) mathematically.

B8. [Compare 9.3.4 and 9.3.5] a) Give the equation if the directrix is $y = 5$ and $e = 3$ using 9.3.5. b) Graph it. c) Suppose we treat $y = 5$ as $y = -(-5)$ and use 9.3.4. Give the equation. d) Graph it. e) How do the two graphs differ? f) Explain the answer to part (e) mathematically.

B9. Equations 9.3.2 and 9.3.4 look just alike except the first "cos" where the second has "sin". State a relevant trig identity of the form "$\sin\theta = \cos(\theta - c)$" and use it to explain why the second graph is just the first graph rotated through 90°.

^^^^The following equations yield hyperbolas on $[0, 2\pi)$. Which angles yield the left branch of the hyperbola?

B10. $r = 1/(1 - 2\cos\theta)$. B11. $r = 5/(1 - 3\cos\theta)$.

B12. For a hyperbola with horizontal axis, give the slopes of the asymptotes if the eccentricity is 2. [$m = \pm 1.7$]

B13. For a hyperbola with horizontal axis, show the slopes of the asymptotes are given by

$$m = \pm\sqrt{e^2 - 1}\ .$$

B14. Let $r = 6/(1 - 2\cos\theta)$. a) Use 9.3.6 to find a. b) Find b and then b/a. c) For which angle θ is the denominator zero? d) What happens, graphically, as θ approaches that value? e) Find tangent of that angle. f) comment on why the answers to (b) and (e) are related.

B15. If the slopes of the asymptotes of a horizontal hyperbola are $\pm 1/4$, give its eccentricity.

B16. If the slopes of the asymptotes of a horizontal hyperbola are $\pm m$, show its eccentricity is $e = \sqrt{(m^2 + 1)}$.

B17. For an ellipse, give the ratio of width to length if the eccentricity is 1/2. [.87]

B18. For an ellipse with eccentricity e, show the ratio of width to length is $\sqrt{(1 - e^2)}$.

B19. If the ratio of width to length of an ellipse is 1/3, give the eccentricity. [.94]

B20. If the ratio of width to length of an ellipse is r, show the eccentricity is $\sqrt{(1 - r^2)}$.

B21. If $e = .1$ and the ellipse closely resembles a circle of radius 5, where is the directrix?

B22. If $e = .01$ and the ellipse closely resembles a circle of radius 3, where is the directrix?

B23. Suppose two ellipses are very similar to a circle with radius 1. One has $e = .02$ and the other has $e = .01$. Compare the locations of their directrixes. [Use the from 9.3.2, or work with 9.3.6. Rough approximations are encouraged.]

B24. Derive 9.3.4, paralleling the steps from which 9.3.2 was derived.

^^^^Sketch the graph, locate the center, and determine the slopes of the asymptotes of:

B25. $r = \dfrac{1}{1 - 1.01\cos\theta}$ [$m = \pm.14$]

B26. $r = \dfrac{1}{1 + 1.2\sin\theta}$ [$m = \pm1.5$]

^^^^Sketch the graph, locate the center, and determine the semi-major axis and the semi-minor axis of

B27. $r = \dfrac{2}{1 - .7\cos\theta}$ [*C*: (2.4, 0)]

B28. $r = \dfrac{5}{1 + .7\sin\theta}$ [*C*: (0, -6.9)]

^^^^Find the rectangular-coordinate equation of a conic section with the given characteristics.
B29. $e = 1/2$, one focus is at the origin, the other focus is on the positive x-axis, and the semi-major axis is 3.
B30. $e = 3$, one focus is at the origin, the other focus is on the negative x-axis, and the semi-major axis is 2.

B31. Suppose you wish to graph $r = f(\theta)$ using a graphics calculator, but it does not have "Polar" mode. You may use "Parametric" mode, in which x and y are entered separately in terms of θ. Because $x = r\cos\theta$ and $y = r\sin\theta$, to graph "$r = f(\theta)$" you may enter:

$$x = f(\theta)\cos\theta. \quad y = f(\theta)\sin\theta.$$

To graph in parametric mode the graph in Example 3, what must you enter in for "x" and what for "y"?

B32. The definition of eccentricity is given in 9.3.1 where ellipses and hyperbolas are defined in terms of a directrix and one focus (instead of the two foci used in Section 9.2). a) Sketch an ellipse with a focus at the origin and vertical directrix at $x = -d$. Label the rectangular coordinate parameters on the sketch. b) Find d in terms of e, a, b, and c.
c) Find e in terms of a, b, and c.

B33. a) Use the definition of e from 9.3.1 and *rectangular* coordinates to set up a rectangular-coordinate equation for a parabola. b) Simplify it.

B34. a) Use the definition of e from 9.3.1, directrix $x = -d$, and *rectangular* coordinates to set up a rectangular-coordinate equation for an ellipse with eccentricity $e < 1$. b) Solve for y. c) Let $e = 1/2$ and $d = 3$ in part (b) and graph the equation.

B35. a) Use the definition of e from 9.3.1, directrix $x = -d$, and *rectangular* coordinates to set up a rectangular-coordinate equation for a hyperbola with eccentricity $e > 1$. b) Solve for y. c) Let $e = 2$ and $d = 3$ in part (b) and graph the equation.

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