PRECALCULUS

Chapters 8 and 9

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CHAPTER 8

Other Topics

Section 8.1. Vectors

Many quantities, such as length and mass, can be described with a single number, but some, such as velocity and force, cannot. We describe them with vectors, which are two or more numbers listed in order. In physics, a vector is equivalent to directed magnitude.

Example 1: A wind is 20 miles per hour from the southwest (Figure 1). Using trigonometry, its easterly component is 20 cos 45° = 14.1 and its northerly component 20 sin 45°, also 14.1. The wind can be described as a vector:
$\mathbf{w} = \langle 20 \cos 45°, 20 \sin 45° \rangle = \langle 14.1, 14.1 \rangle$.

Wind velocity can not be described by a single number. It can be described by a magnitude (speed) and direction, or, equivalently, as a vector by giving its two components in primary directions. (Speed is not velocity because velocity also refers to direction.)

A vector can be represented by any line segment of the proper length and direction (Figure 1), and all such line segments are said to be equivalent.

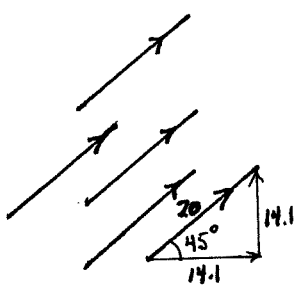

Figure 1: A wind toward the northeast.

Notation: Vectors are often denoted by boldface letters such as **v** and **w**. If **v** has two components, it may be written as an ordered pair: $\mathbf{v} = \langle v_1, v_2 \rangle$. We use pointed brackets to distinguish vectors from points. Some texts just use parentheses: $\mathbf{v} = (v_1, v_2)$.

Example 2: A wind vector is $\mathbf{w} = \langle 50, 20 \rangle$ (Figure 2). What is its magnitude (speed) and direction?

Using the Pythagorean Theorem, the magnitude, $|\mathbf{w}|$, satisfies

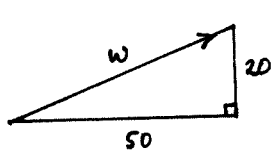

Figure 2: $\mathbf{w} = \langle 50, 20 \rangle$

$$|\mathbf{w}|^2 = 50^2 + 20^2.$$
$$|\mathbf{w}| = 53.85\ .$$

Using trigonometry, the counterclockwise angle θ with the horizontal (the angle north of due east) satisfies

$$\tan\theta = 20/50\ .$$
$$\theta = 21.8^\circ.$$

Therefore the vector $\langle 50, 20\rangle$ could also be described by "magnitude 53.85 at angle 21.8°" (counterclockwise from the positive x-axis).

Angles for vectors are described with respect to the x-axis. However, directions on the ground are usually described by "bearing" which is, in this case, the angle with due north, which is $90° - \theta = 68.2°$. The wind is 53.85 miles per hour in the direction N 68.2° E (which is 21.8° north of east).

Figure 3 relates the components to the length and direction.

<u>Definition 8.1.1</u>: The <u>magnitude</u> or <u>length</u> of a vector $\mathbf{v} = \langle a, b\rangle$ is denoted $|\mathbf{v}|$. From the Pythagorean Theorem (Figure 3),

$$|\mathbf{v}| = \sqrt{a^2 + b^2}\ .$$

The angle with the x-axis, θ, satisfies

$$\tan\theta = b/a, \text{ if } a \neq 0.$$

In terms of its magnitude (length) and angle,

$$\mathbf{v} = \langle |\mathbf{v}|\cos\theta, |\mathbf{v}|\sin\theta\rangle.$$

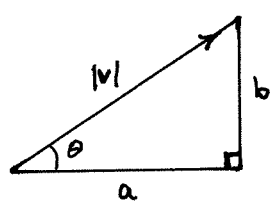

Figure 3: $\mathbf{v} = \langle a, b\rangle$

<u>Definition 8.1.2 (Vector addition and multiplication of a vector by a number)</u>: Adding vectors is done componentwise.

Let $\mathbf{v} = \langle v_1, v_2\rangle$ and $\mathbf{w} = \langle w_1, w_2\rangle$. Then

$$\mathbf{v} + \mathbf{w} = \langle v_1+w_1, v_2+w_2\rangle.$$

Multiplying a vector by a number is done componentwise.

$$c\langle v_1, v_2\rangle = \langle cv_1, cv_2\rangle\ .$$

Example 3: Find $\langle 3, 10\rangle + \langle 2, 4\rangle$.

$\langle 3, 10\rangle + \langle 2, 4\rangle = \langle 5, 14\rangle$

Solve for x: $\langle 1, 5\rangle + \langle x, 2\rangle = \langle 9, 7\rangle$.

$1 + x = 9.\ x = 8.$

A wind vector is $\mathbf{w} = \langle 3, 10\rangle$. If it doubles in magnitude, what will it be?

It will be $2\langle 3, 10\rangle = \langle 6, 20\rangle$.

Find $\mathbf{v} + \mathbf{w}$ if $|\mathbf{v}| = 7$ and its angle with the positive x-direction is 30° and $\mathbf{w} = \langle 1.4, -1\rangle$.

First write $\mathbf{v}$ in component form. $\mathbf{v} = \langle 7\cos 30°, 7\sin 30°\rangle = \langle 6.06, 3.5\ \rangle$.

Therefore $\mathbf{v} + \mathbf{w} = \langle 6.06, 3.5\ \rangle + \langle 1.4, -1\rangle = \langle 7.46, 2.5\rangle$.

Physics tells us that velocity vectors may be added "componentwise". For example, when a plane flies through the air, the velocity of the air is added to the velocity of the plane through the air to find the velocity of the plane with respect to the ground.

Example 4: Suppose a plane flies at 150 miles per hour relative to the wind with its nose at bearing N 30° E (Figure 4, **v**). The wind is 40 miles per hour from S 70° W. (So the wind vector is pointing N 70° E, **w**) What is the velocity of the plane relative to the ground?

Relative to the wind, the velocity vector of the plane (which has direction 60° from the horizontal) is

$$\mathbf{v} = \langle 150 \cos 60^\circ, 150 \sin 60^\circ \rangle$$
$$= \langle 75, 129.9 \rangle.$$

The velocity vector of the wind (which has direction 20° from the horizontal) is

$$\mathbf{w} = \langle 40 \cos 20^\circ, 40 \sin 20^\circ \rangle$$
$$= \langle 37.6, 13.7 \rangle.$$

Adding componentwise, the sum is

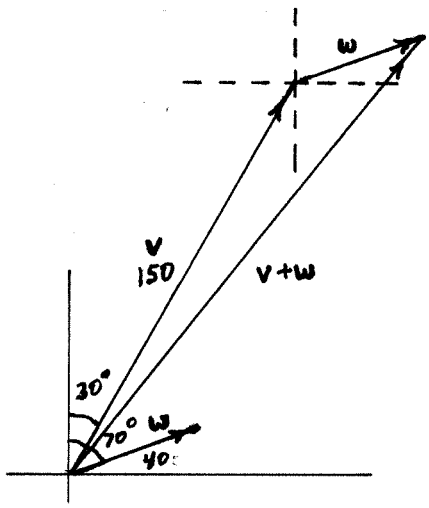

Figure 4: Velocity **v**, wind velocity **w**, and ground velocity, **v**+**w**.

$$\mathbf{v} + \mathbf{w} = \langle 75, 129.6 \rangle + \langle 37.6, 13.7 \rangle$$
$$= \langle 75+37.6, 129.6+13.7 \rangle$$
$$= \langle 112.6, 143.3 \rangle.$$

This is the ground velocity, which has

$$\text{ground speed} = \sqrt{112.6^2 + 143.3^2} = 182.2 \text{ (miles per hour)}$$

and direction θ satisfying

$$\tan\theta = \frac{143.3}{112.6} = 1.273.$$

Because θ is in the first quadrant,

$$\theta = 51.8^\circ.$$

The direction is 51.8° north of east, which may be written "N 38.2 E."

<u>The Parallelogram Law</u>. Adding vectors has a simple geometric representation (Figures 5A and 5B). Technically, vectors do not have locations (just magnitude and direction), but they can be represented anywhere. To represent the sum of **v** and **w**, represent **v** with its "tail" at the origin, and **w** with its tail at the "head" of **v** (Figure 5B). Then **v**+**w** is represented as the vector from the tail of **v** to the head of **w**. Figure 5B puts **w** in both places and shows why this method of addition is called the "parallelogram law" (see Problem B21 for subtraction).

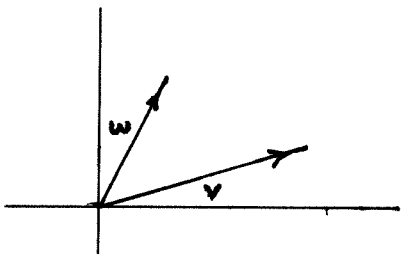

Figure 5A
v, w
$\langle v_1, v_2\rangle, \langle w_1, w_2\rangle$

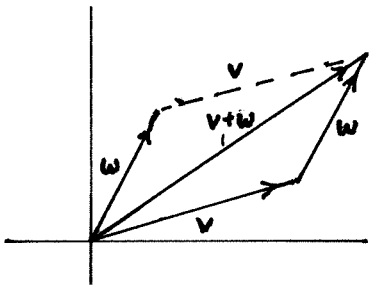

Figure 5B
v, w, v+w
$\langle v_1, v_2\rangle + \langle w_1, w_2\rangle = \langle v_1+w_1, v_2+w_2\rangle$

<u>Theorem 8.1.3</u>: A) $|c\mathbf{v}| = |c||\mathbf{v}|$, where c is any constant.
B) $|\mathbf{v} + \mathbf{w}| \le |\mathbf{v}| + |\mathbf{w}|$, "the triangle inequality"

Figure 5B illustrates the triangle inequality. "A line is the shortest distance between two points." The distance from the origin to the tip of **v**+**w** is |**v**+**w**|. Theorem 8.1.3B says this is less than or equal to the sum of the lengths of **v** and **w**.

In physics, **F** = ***ma***, that is, <u>f</u>orce is <u>m</u>ass times <u>a</u>cceleration. Both force and acceleration are vectors because they have direction as well as magnitude. If an object is at rest, the total force on it is **0**, the zero vector.

Example 5: A cart on an plane inclined 20° is kept from sliding down the plane by a cable pulling along the plane (Figure 6A). If the force of gravity on the cart is 500 newtons, what is the force of the cable on the cart?

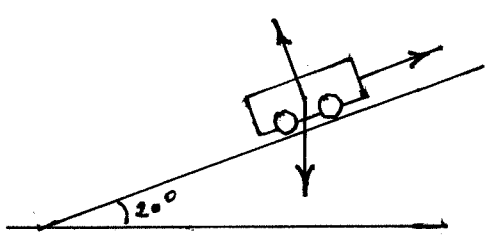

Figure 6A: A cart, gravity and a cable.

First pick a convenient coordinate system. We will pick the x-direction to be along the plane and the y-direction to be perpendicular to the plane (Figure 6B).

The force, $\mathbf{F}_c$, of the cable is unknown, but it is entirely along the plane, so the component perpendicular to the plane is 0. Therefore,

$\mathbf{F}_c = \langle c, 0\rangle$, for some c.

Using trigonometry, the force of gravity can be "resolved" into components.

$\mathbf{F}_g = \langle -500 \sin 20^\circ, -500 \cos 20^\circ\rangle$

(The minus signs are because the components are to the left and down.)

The third force is the plane pushing on the cart, perpendicular to the incline, so the x-component of that force is zero. It is $\langle 0, d\rangle$ for some d.

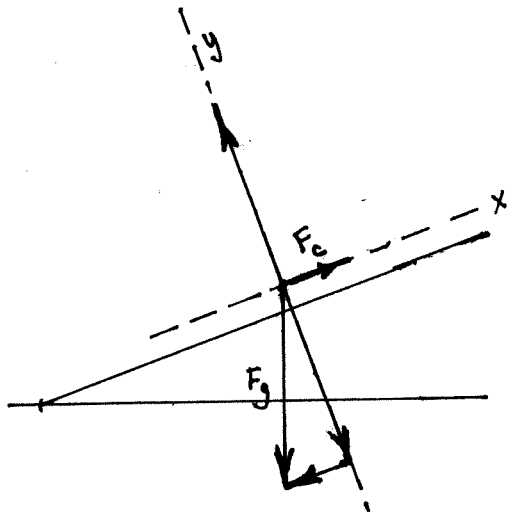

Figure 6B: An axis system, and force vectors resolved into components.

Because the cart is at rest, the sum of the forces on it must be the zero vector.

$$\langle c, 0\rangle + \langle -500 \sin 20^\circ, -500 \cos 20^\circ\rangle + \langle 0, d\rangle = \langle 0, 0\rangle .$$

along the plane + into the plane + out of the plane = the zero vector.

So, adding all the x-components,

$$c + -500 \sin 20^\circ + 0 = 0 .$$

Solving,

$$c = 500 \sin 20^\circ = 171.0 \text{ (newtons)}.$$

The tension in the cable is 171 newtons (see problem A15 for d).

Example 6: A 20 pound object is suspended by two cables attached to A and B as in Figure 7A. What are the forces exerted by the cables?

There are three forces, as represented in Figure 7B.

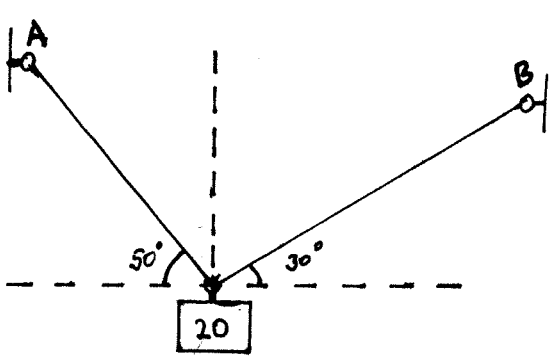

Figure 7A: An object suspended by two cables.

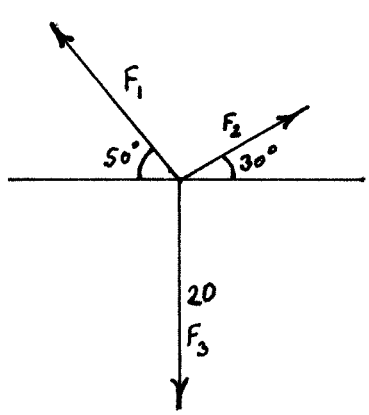

Figure 7B: Three forces on an object.

Because the object is at rest, it has no acceleration and the total force on it must be **0**, the zero vector. Therefore, $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$, where

$\mathbf{F}_1 = \langle -|\mathbf{F}_1| \cos 50^\circ, |\mathbf{F}_1| \sin 50^\circ \rangle$

$\mathbf{F}_2 = \langle |\mathbf{F}_2| \cos 30^\circ, |\mathbf{F}_2| \sin 30^\circ \rangle$

$\mathbf{F}_3 = \langle 0, -20 \rangle$

Adding the x-components,

$$-|\mathbf{F}_1| \cos 50^\circ + |\mathbf{F}_2| \cos 30^\circ + 0 = 0 .$$

(*) $$|\mathbf{F}_2| = |\mathbf{F}_1| \frac{\cos 50^\circ}{\cos 30^\circ} .$$

Adding the y-components,

$$|\mathbf{F}_1| \sin 50^\circ + |\mathbf{F}_2| \sin 30^\circ - 20 = 0 .$$

Now there are two equations and two unknowns, $|\mathbf{F}_1|$ and $|\mathbf{F}_2|$. Use equation (*) to replace $|\mathbf{F}_2|$ in the second equation:

$$|\mathbf{F}_1| \sin 50^\circ + |\mathbf{F}_1| \frac{\cos 50^\circ}{\cos 30^\circ} \sin 30^\circ = 20 .$$

Now there is only one unknown, $|\mathbf{F}_1|$.

$$|\mathbf{F}_1| \left(\sin 50^\circ + \frac{(\cos 50^\circ)(\sin 30^\circ)}{\cos 30^\circ} \right) = 20 .$$

$$|\mathbf{F}_1| = 17.59 \text{ (pounds)}.$$

If we want $\mathbf{F}_1$ in components, it is

$$\mathbf{F}_1 = \langle -|\mathbf{F}_1| \cos 50^\circ, |\mathbf{F}_1| \sin 50^\circ \rangle = \langle 11.31, 13.47 \rangle.$$

To find $|\mathbf{F}_2|$, use (*).

$$|\mathbf{F}_2| = 13.06, \text{ and } \mathbf{F}_2 = \langle 11.31, 6.53 \rangle .$$

<u>Standard Unit Vectors</u>. "Unit" vectors are vectors of length one. The vectors $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ are the "standard unit vectors." They are often given a special notation:

(8.1.4) $$\mathbf{i} = \langle 1, 0 \rangle \text{ and } \mathbf{j} = \langle 0, 1 \rangle.$$

Note that **i** and **j** are in boldface type. For handwriting, an arrow over the top of "i" and "j" may serve instead of boldface. Any two-dimensional vector can be written as a linear combination of **i** and **j**.

(8.1.5) $$\langle a, b \rangle = a\mathbf{i} + b\mathbf{j}.$$

Example 7: $\langle 3, 5 \rangle = 3\mathbf{i} + 5\mathbf{j}$.
$5\mathbf{i} - \mathbf{j} = \langle 5, -1 \rangle$.

If the vectors have three dimensions, then "**k**" is used for the third vector:

$\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$, and $\mathbf{k} = \langle 0, 0, 1\rangle$.

Conclusion: Vectors describe quantities, such as force and velocity, with more than one dimension.

Terms. Vector, magnitude, length, direction.

Exercises for Section 8.1, "Vectors."

^^^^ Evaluate

A1. $\langle 3, 5\rangle + \langle 1, 4\rangle$ A2. $\langle 4, 1\rangle + \langle 5, 2\rangle$ A3. $\langle 1, 4.3\rangle + \langle 2.5, 3.7\rangle$
A4. $\langle 1.9, 1.2\rangle + \langle 4.3, -2.5\rangle$ A5. $2\langle 3, 4\rangle$ A6. $3\langle 1, 2\rangle$

^^^^ Solve for $\langle x, y\rangle$.

A7. $\langle 1, 5\rangle + \langle x, y\rangle = \langle 3, 9\rangle$ A8. $\langle 4, 7\rangle + \langle x, y\rangle = \langle 6, 12\rangle$
A9. $\langle x, y\rangle + \langle 4, 9\rangle = \langle 3, 12\rangle$ A10. $\langle x, y\rangle + \langle -1, 6\rangle = \langle 3, 8\rangle$

A11. Find $\langle 1, 2, 3\rangle + \langle 4, 5, 6\rangle$. A12. Find $2\langle 1, 2, 3\rangle - \langle 4, 5, 6\rangle$.

A13. The airspeed vector of a plane is $\langle 125, 20\rangle$ and the windspeed vector is $\langle -20, -5\rangle$. What is the groundspeed vector?
A14. The airspeed vector of a plane is $\langle 540, -40\rangle$ and the windspeed vector is $\langle -80, -30\rangle$. What is the groundspeed vector?

A15. In Example 5, find d. [470]

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B1. Draw a picture to illustrate the relationship between a vector and its length and direction.

B2. Let the vector **v** be of length r with direction θ (counterclockwise) from the positive x-direction. Give the vector **v**, written as an ordered pair.

^^^^ Find the magnitude and direction of the vector.

B3. $\langle 5, 2\rangle$ B4. $\langle 4, -1\rangle$ B5. $\langle -8, 6\rangle$ B6. $\langle 87, 50\rangle$

^^^^ Find the vector as an ordered pair.

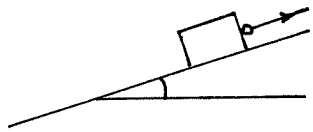
B7. Direction 30°, length 10. B8. Direction 45°, length 100
B9. Direction 150°, length 20. B10. Direction 100°, length 5.

B11. Suppose a plane flies at 550 miles per hour at bearing S 70° W relative to the wind. The wind is 60 miles per hour from N 80° W. (So the wind vector is pointing S 80° E.) What is the velocity of the plane relative to the ground?

B12. A plane flies with its nose headed northeast with airspeed 525 miles per hour. The wind is from N 60° W at 65 miles per hour. What is the groundspeed and actual direction of flight?

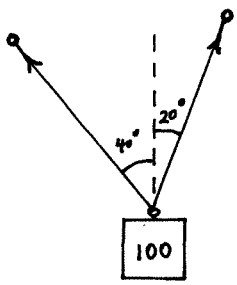
B13. (See the figure.) A 400-pound weight is held motionless on a (frictionless) inclined plane by a cable pulling along the plane, which is at 15° from horizontal. What is the tension in the cable?

B14. (See the figure.) A 250-pound weight is held motionless on a (frictionless) inclined plane by a cable pulling along the plane, which is at 22° from horizontal. What is the tension in the cable?

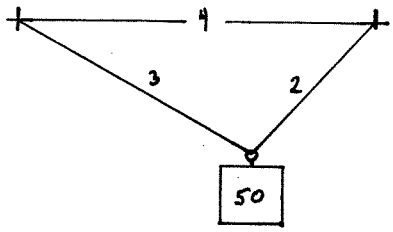


For B13 and B14

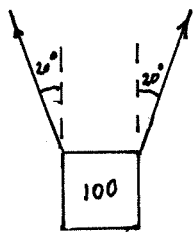
B15. (See the figure.) A 100-pound weight is supported by two cables, one at 40° from vertical, and the other at 20° from vertical. Find the tension in the cables.



For B15



For B16

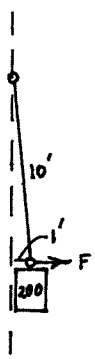


For B17

B16. (See the figure.) A fifty-pound weight is supported between two posts 4 feet apart by cables strung from the same height, one 2 feet long and the other 3 feet long. Find the tension in the cables.

B17. (See the picture.) Two people lift a 100-pound basket by pulling upwards (and slightly outward) on its handles. If each pulls at angle 20° from vertical, how much force does each person use?

B18. (See the figure.) A 200-pound weight is suspended from a 10-foot cable. How much force is required to pull it one foot to the side?



For B18

B19. The <u>dot product</u> of vectors $\mathbf{v} = \langle a, b\rangle$ and $\mathbf{w} = \langle c, d\rangle$ is denoted $\mathbf{v}\bullet\mathbf{w}$ (read "v dot w") and given by

(8.1.6) $\mathbf{v}\bullet\mathbf{w} = ac + bd.$

a) Find $\langle 3, 2\rangle\bullet\langle 4, 10\rangle$.
b) Find $\langle 1, 0\rangle\bullet\langle 0, 1\rangle$.
c) Theorem: $\mathbf{v}\bullet\mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Find the angle between $\langle 3, 1\rangle$ and $\langle 2, 4\rangle$.
d) Find the angle between $\langle 2, 1\rangle$ and $\langle -1, 2\rangle$.
e) If two vectors are perpendicular, their dot product is zero. Why?
f) Find a vector perpendicular to $\langle 1, 5\rangle$.
g) [To prove the theorem] Express the square of the distance between the points (a, b) and (c, d). Then treat the points as tips of the vectors $\mathbf{v}$ and $\mathbf{w}$ and express the square of the distance between them using the Law of Cosines, recalling $|\mathbf{v}|$ and $|\mathbf{w}|$ are the lengths. Equate the two squares of distances and simplify to obtain the theorem.

B20. [Unit vectors] a) Prove "If **v** is a non-zero vector, then **v**/|**v**| is a unit vector."
b) Prove: "Any vector can be written as a constant times a unit vector."
c) Write ⟨2, 3⟩ as a constant times a unit vector.

B21. Subtraction of vectors has a "parallelogram law" type picture. a) Sketch a picture like Figure 5 to illustrate **v** and **w**. On it sketch -**w**. Treat the vector **v**-**w** as **v**+(-**w**) and sketch it.
b) Find the vector **v**-**w** in the parallelogram.
c) Note how the picture illustrates **v** = (**v**-**w**)+**w**.

B22. [Vectors that are not from physics.] A store sells medium, large, and extra large shakes. To describe the number of shakes sold each hour we might use a vector with three components, one for each size of shake. The vector ⟨2, 1, 5⟩ would refer to 2 medium shakes, 1 large shake, and 5 extra large shakes. Of course, we must agree on the order in which the numbers are listed. Suppose medium shakes cost 90 cents, large shakes cost 1.10, and extra large shakes cost 1.40. a) Find the total cost of ⟨2, 1, 5⟩.
b) Find the total cost of ⟨a, b, c⟩.
c) See the definition of "dot product" in B19 (Imagine it extended to three components). The answer to part (b) is the dot product of which two vectors?

B23. [Vectors that are not from physics.] A researcher studies the effectiveness of a new treatment. Ten people are in the study, six of whom are treated and four of whom are "controls" and not treated. The ten people are listed in order and the corresponding treatment vector has a "1" if that person was treated and a "0" if not treated. The treatment vector is
⟨0, 1, 1, 0, 1, 1, 1, 0, 0, 1⟩.
The outcomes are scored on a scale of zero to ten. The outcome vector is
⟨3, 5, 8, 4, 7, 2, 9, 5, 2, 7⟩.
a) Find the averages of the outcomes for treated people and for people who were not treated. Which average is higher?
b) See the definition of "dot product" in B19 (extend the definition to ten components). Write the average outcome for treated people in terms of the dot product of the two vectors.

Section 8.2. Complex Numbers

Complex numbers arise in the context of quadratic equations such as "$x^2 = -4$" and "$x^2 + 2x + 2 = 0$." Since $x^2 \geq 0$ for all real-valued x, the equation "$x^2 = -4$" cannot have any real-valued solutions. The usual way to solve "$x^2 = -4$" would be to take the square root, but negative numbers such as -4 do not have real-valued square roots. Similarly, the usual way to solve a quadratic equation such as "$x^2 + 2x + 2 = 0$" is to use the Quadratic Formula which yields

$$\frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2}$$

Again, the square root of a negative number appears.

Since the square roots of negative numbers are not "real" numbers, one reasonable approach to solving these two equations is to say they have no solutions. Mathematicians accepted this as certainly true and completely obvious for many centuries.

But, it is equally obvious that you cannot take 7 stones away from a pile of 5 stones. Unfortunately, "obvious" facts have a habit of obstructing the development of mathematics. But, as you know, when negative numbers were finally invented, they turned out to have many useful but unsuspected applications (although not to taking 7 stones from a pile of 5 stones, which cannot be done). For example, now we are all familiar with negative numbers in the contexts of temperatures, money, checking accounts, and credit-card balances.

Another, bold, approach to solving these quadratic equations is to create solutions by the simple device of inventing a solution to the equation "$x^2 = -1$." Call that invented solution "i" (pronounced as it looks, "eye"), so $i^2 = -1$. Assume that the usual properties of combinations of arithmetic operations apply to this new type of number. Then $(2i)^2 = 2^2 i^2 = 4i^2 = 4(-1) = -4$. So $2i$ will be a solution to the equation "$x^2 = -4$." Similarly, $-2i$ will be a second solution.

Reconsider the "solution" to the equation "$x^2 + 2x + 2 = 0$" provided by the Quadratic Formula. Using our new-found square root of negative four and the usual arithmetic operations on this new type of number,

$$\frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i.$$

With this abstract approach all quadratic equations that have no real-valued solutions will have two "complex-valued" solutions.

You may be wondering if this approach yields anything important, or if it is just a way to create a useless solution to a useless equation. Well, you can probably guess the answer. The reason math books have section on complex numbers is, of course,

that in some subjects they turn out to be extremely useful. For example, they are an essential fixture of electrical engineering (where our "i" is called "j", because they need to use the symbol "i" for something else). They are critical to the study of laser physics. In fact, in any subject where electro-magnetic waves occur, the theory is likely to use complex numbers. And, in mathematics, complex numbers have remarkable connections to trigonometry and rotations of two-dimensional figures. Like the applications of negative numbers, the applications of complex numbers have appeared in unanticipated places. For example, they are now used in the generation of fractals. Fractals are fantastically complicated images that arise from remarkably simple computer instructions (Figure 1, problem B29). Because of their promise for encoding a large amount of visual information in a few lines of computer code (and because they can be entrancingly beautiful), they are currently receiving much attention.

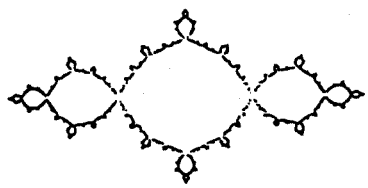


Figure 1: A Fractal image.

We cannot develop the subjects of electrical engineering, laser physics, or fractals here. But, every subject has to start somewhere. Complex numbers start with i.

<u>Definition 8.2.1 (Complex Numbers)</u>: There is a solution to "$x^2 = -1$" called i. The <u>complex numbers</u> consist of all numbers of the form "$a + bi$" where "a" and "b" are real numbers.

The use of the letters "a" and "b" to represent real numbers in this form is traditional. If we wish to denote a complex number by a single letter, we will use "z" or "w", never "a" or "b". So, in traditional notation, $z = a + bi$.

Complex numbers can be added, subtracted, multiplied, and divided according the usual properties of these operations on real numbers.

Example 1: Some complex numbers in "$a + bi$ form" are

$1 + i$ [$a = 1$ and $b = 1$],
$7 - 5.67i$ [$a = 7$ and $b = -5.67$],
$-1/2 - i/4$ [$a = -1/2$ and $b = -1/4$.],

Other complex numbers in "$a + bi$ form" are

i [$a = 0$ and $b = 1$],
3 [$a = 3$ and $b = 0$], and
0 [$a = 0$ and $b = 0$].

The complex numbers include the real numbers. All real numbers are complex numbers, but not all complex numbers are real numbers.

The solution to "$x^2 = -1$," i, is often said to be an <u>imaginary</u> number. With a good imagination, you can make it "real" to you. Just work with it enough and, like other abstractions, it will take on reality as its usefulness becomes apparent. Because

$i^2 = -1$, i is often called "the square root of negative one."

Numbers of the form "bi" where "b" is real are said to be pure imaginary numbers. In the complex number "$a + bi$," "a" is the real part and "bi" is the imaginary part.

Two complex numbers "$a + bi$" and "$c + di$" are said to be equal if and only if $a = c$ and $b = d$, that is, if their real parts are the same and their imaginary parts are the same.

Real numbers can be located on a one-dimensional number line. Complex numbers cannot. They are two-dimensional and require two dimensions to locate.

To graph a complex number let the horizontal axis be the real axis and the vertical axis be the imaginary axis. Use a square scale. Plot $z = a + bi$ in the usual position of the ordered pair (a, b) (Figure 2). When the plane is used to plot complex numbers in this manner it is called the complex plane. Figure 3 plots a few complex numbers.

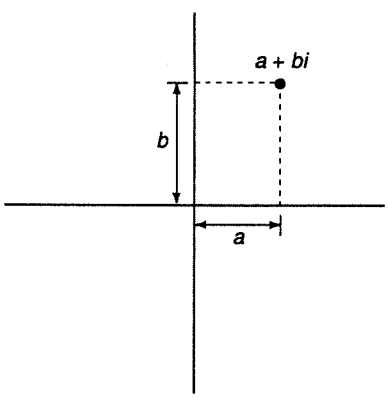


Figure 2: $a + bi$
[-5, 5] by [-5, 5].

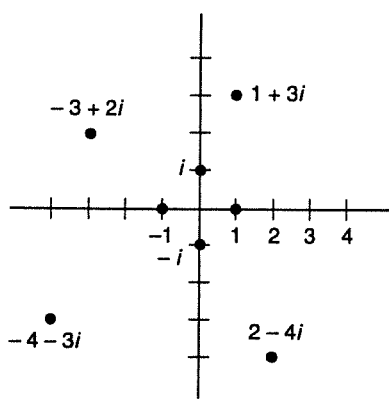


Figure 3: A few complex numbers.
[-5, 5] by [-5, 5]

The idea that complex numbers consist of two distinct parts is critical to working with them. For example, to add or subtract two complex numbers, add or subtract their real and imaginary parts separately. There is no trick to adding or subtracting complex numbers. Simply consolidate like terms.

Example 2: $(2 + i) + (1 + 3i)$
$= 2 + 1 + i + 3i$
$= 3 + 4i$ (Figure 4).

Subtraction is done similarly:
$(2 + i) - (1 + 3i) = 2 - 1 + i - 3i = 1 - 2i$.

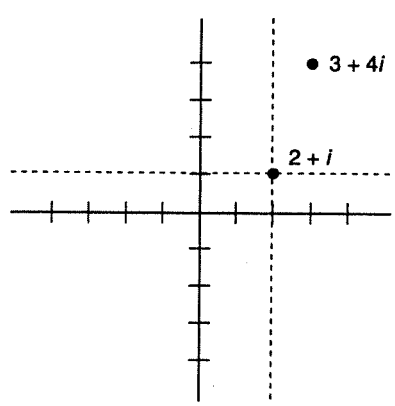


Figure 4: Adding complex numbers.
$(2+i) + (1+3i) = 3+4i$.
[-5, 5] by [-5, 5].

The abstract form of this simple idea is next.

<u>Property 8.2.2 (Addition and Subtraction)</u>:
$(a + bi) + (c + di) = (a + c) + (b + d)i.$
$(a + bi) - (c + di) = (a - c) + (b - d)i.$

Graphically, adding has two common interpretations. To add $z + w$, plot z and then imagine the location of z to be a new origin (Figure 4). Then plot w relative to the new origin. That will be the position of $z + w$ relative to the original origin. This is much like adding numbers on the real number line, where, to add 2 + 3 you may find position "2" and treat it as a new origin. Then the position "3" relative to that new origin is position "5" relative to the original origin.

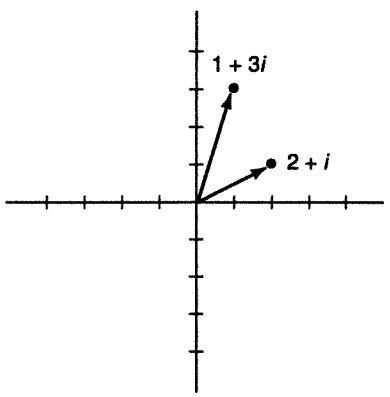


Figure 5: $1 + 3i$ and $2 + i$. [-5, 5] by [-5, 5].

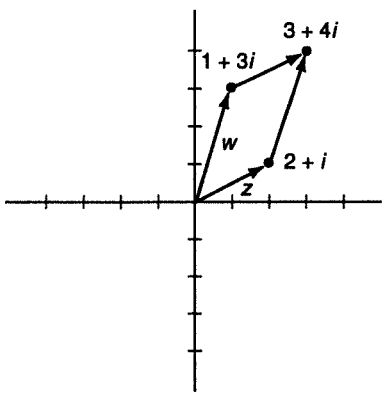


Figure 6: The sum of $1 + 3i$ and $2 + i$. [-5, 5] by [-5, 5]

This addition may be visualized another way. Plot both z and w and draw arrows (vectors) from the origin to their locations (Figure 5). Then slide the vector for w so its tail is at z (Figure 6). Then the head of the vector w will be at $z + w$. This is the so-called "parallelogram" approach because the construction can create a parallelogram (Figure 6). By the Commutative Property, $z + w = w + z$, and the drawing for these two expressions creates opposite sides of a parallelogram.

Graphically, the function $f(z) = z + w$ ("add w") produces a translation (that is, a location shift).
Points are translated the distance and direction of w from the origin.

<u>**Multiplication**</u>. The key to multiplication of complex numbers is that the Distributive Property and the Extended Distributive Property still hold. Therefore, all you have to do is "multiply out" the expressions, consolidate like terms, and remember that $i^2 = -1$.

Example 3: $(3 + 2i)(7 + 4i) = 21 + 12i + 14i + 8i^2$ [using FOIL, 3.4.2]
[Recall that $i^2 = -1$, so $8i^2 = -8$]
$= 21 - 8 + (12 + 14)i$
$= 13 + 26i.$

Example 4: $(1 - i)(3 + 2i) = 3 + 2i - 3i - 2i^2$
$= 3 + 2 + (2 - 3)i$
$= 5 - i.$

This process can be expressed abstractly.

$$\begin{array}{r} a + bi \\ \underline{c + di} \\ adi + bdi^2 \\ \underline{ac + bci\qquad} \\ ac + adi + bci + bdi^2 \end{array} = (ac - bd) + (ad + bc)i.$$

<u>Property 8.2.3</u>: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$

Rather than memorize this identity with all its dummy variables, simply use the Extended Distributive Property (FOIL) to multiply complex numbers.

<u>Property 8.2.4</u>: If $p \geq 0$, the solution to $z^2 = -p$ is $z = \pm i\sqrt{p}$.

By the multiplication property 8.2.3,

$$(i\sqrt{p})^2 = (i\sqrt{p})(i\sqrt{p}) = i^2(\sqrt{p})^2 = -p.$$

So $i\sqrt{p}$ is a solution to "$x^2 = -p$." $-i\sqrt{p}$ is another.

Square roots of negative numbers may appear in the Quadratic Formula.

Example 5: Solve $x^2 + 4x + 8 = 0$.
Use the Quadratic Formula.

$$x^2 + 4x + 8 = 0 \quad \text{iff} \quad x = \frac{-4 \pm \sqrt{4^2 - 4(1)(8)}}{2(1)}$$

$$\text{iff } x = \frac{-4 \pm \sqrt{-16}}{2} = \frac{-4 \pm i\sqrt{16}}{2} \qquad \text{[by 8.2.4]}$$

$$\text{iff } x = \frac{-4 \pm 4i}{2}$$

$$\text{iff } x = -2 \pm 2i.$$

The answer is in "$a + bi$" form. The last line follows by straightforward division. As expected, multiplication or division of a complex number by a real number, c, simply multiplies or divides both parts. This is stated abstractly next.

8.2.5 (Corollary to 8.2.3): $c(a + bi) = ca + cbi$

$(a + bi)/c = a/c + (b/c)i$, if $c \neq 0$.

Example 6: $3(1 + i) = 3 + 3i$ (Figure 7).

$(-2 + 4i)/4 = -1/2 + i$, where $a = -1/2$ and $b = 1$ (Figure 7).

Figure 7 illustrates that multiplying a complex number by a real number changes its distance from the origin, but not its direction from the origin. Multiplying by $c > 1$ expands the point away from the origin by a factor of c. Multiplying by c between 0 and 1 contracts the point toward the origin.

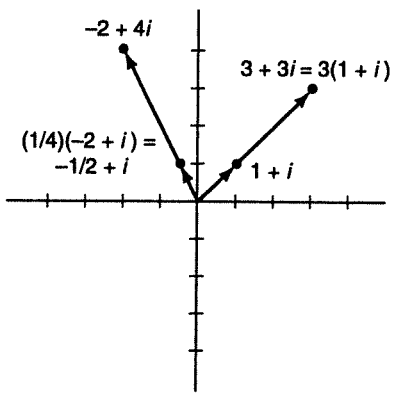


Figure 7: Multiplying by a real number.
[-5, 5] by [-5, 5].

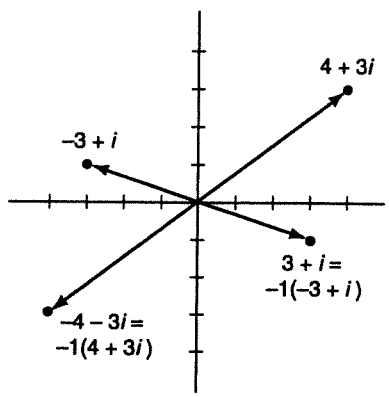


Figure 8: Multiplying by -1.
[-5, 5] by [-5, 5].

Multiplying by -1 reverses the direction of the point from the origin.

Example 7: $-1(4 + 3i) = -4 - 3i$ (Figure 8).

$-1(3 - i) = -3 + i$ (Figure 8)

Subtraction can be regarded as addition of the negative.

$$z - w = z + (-1)w = z + (-w).$$

That is, we can distribute the minus sign like usual. Graphically, we may interpret $z - w$, using the parallelogram idea, as $z + (-w)$, where $-w$ has the length of w but the opposite direction.

Example 8: $(1 + 2i) - (3 + i) = -2 + i$ (Figure 9).

$(1 + 2i) - (-2 - 2i) = 3 + 4i$ (Figure 10).

For the first example, Figure 9 plots $1+2i$, $3+i$, and $-(3+i)$. Then the parallelogram process is used to subtract $3+i$ by *adding* $-(3+i)$.

For the second example, to subtract $-2-2i$, plot it and then $-(-2-2i)$. Then *add* $-(-2-2i)$.

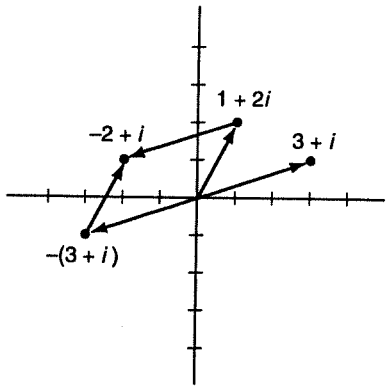


Figure 9:
$(1+2i) - (3+i) = -2+i$
[-5, 5] by [-5, 5].

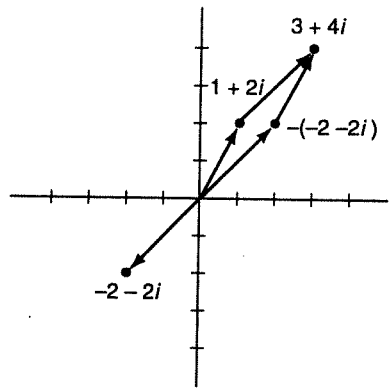


Figure 10:
$(1+2i) - (-2-2i) = 3+4i$.
[-5, 5] by [-5, 5].

So far we have seen that addition, subtraction, and multiplication by a real number are straightforward to interpret in the complex plane. The unexpected properties lie ahead.

Multiplication and Rotation. In the complex plane multiplication has a truly remarkable interpretation which is not at all evident from the algebraic form of multiplication expressed in Property 8.2.3. We will show that multiplication by a complex number corresponds to a *rotation* and a *contraction* or *expansion* toward or away from the origin. Before we formalize this thought, consider a simple example.

Example 9: Consider multiplication by i. Multiply several complex numbers by i and inspect their graphs to see what the graphical effect of multiplying by i is.

Any complex numbers will do. For instance, consider 2 [P], $4i$ [Q], and $-3 - i$ [S] (Figure 11).

$2(i) = 2i$ [P']
$(4i)i = 4i^2 = -4$ [Q']
$(-3 - i)i = -3i - i^2 = 1 - i$ [S']

Every time the product is the same distance away from the origin as the original number, but rotated $\pi/2$ (90°) counterclockwise. **Multiplying any number by i rotates the number by $\pi/2$ (90°).**

This is related to why $i^2 = -1$. Think of i^2 as i times i. Rotating i $\pi/2$ yields -1. $i^2 = -1$.

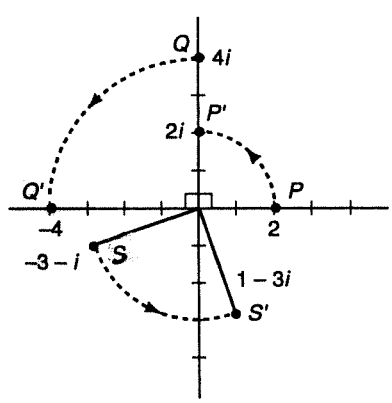


Figure 11: Complex numbers multiplied by i.

Trigonometric Form. To see why multiplication and rotation are related, we use an alternative method to describe the position of a complex number in the complex plane. The idea is to describe the point by its distance from the origin and its angle with the positive real axis (Figure 12). Because angles and distances are used, complex numbers described this way are said to be in "trigonometric form" which is also known as "polar form" (the pole is the origin).

The distance from $z = a + bi$ to the origin is called its modulus (or, its absolute value), which is often denoted "$|z|$" or "r" (for "radius," when r is positive). By the Pythagorean Theorem,

(8.2.6) $$|z| = \sqrt{a^2 + b^2} = r.$$

$z = 0$ if and only if $r = 0$.

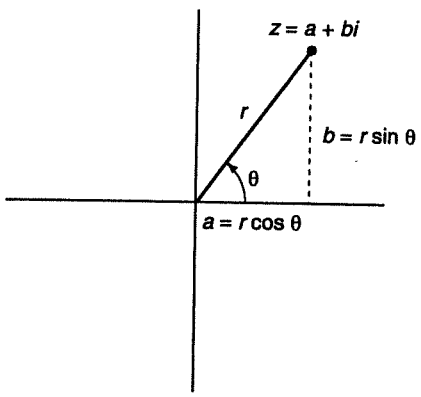


Figure 12: r and θ for "trigonometric form" of complex numbers. [Memorize this picture. All of 8.2.7 follows from it.]

Example 10: $|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$.
$|-1 - 3i| = \sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$.
$|2.34i| = 2.34$.
$|-5.67| = 5.67$.

The angle the line from the origin through z makes with the positive real axis is called the argument of z ("arg z"). Being an angle, the argument of z is often denoted by "θ" (Figure 12). All the results in the next list follow immediately from the picture. You do not have to memorize all these facts -- just remember the picture. "A picture is worth several complex-number facts."

(8.2.7) Let $z = a + bi$. Then
arg $z = \theta$ [which is simply the usual notation].
$\tan\theta = b/a$, if $a \neq 0$. θ is not defined if $z = 0$.
$\cos\theta = a/r$ and $\sin\theta = b/r$, if $z \neq 0$ (that is, if $r \neq 0$).
$a = r\cos\theta$ and $b = r\sin\theta$, if $z \neq 0$.
If $a > 0$, θ is in the first or fourth quadrant.
If $a < 0$, θ is in the second or third quadrant.
If $a = 0$ and $b > 0$, $\theta = \pi/2$.
If $a = 0$ and $b < 0$, $\theta = -\pi/2$.

The second last line mentions "$\pi/2$" rather than "90°" because trigonometric form is usually expressed using radian measure (which has advantages over degree

measure when applied to advanced material). Actually, θ is not uniquely determined, since adding 2π (360°) to any argument yields another coterminal angle. But, whenever convenient, we prefer to use θ between $-\pi$ and π, or, sometimes, between 0 and 2π.

Example 11: $\arg(1 + i) = \pi/4$.

To see this, think of a picture of the location of $1 + i$ in the complex plane (Figure 13). From the picture (or 8.2.7), $\tan\theta = b/a = 1/1 = 1$. So, $\theta = \tan^{-1}(1) = \pi/4$.

$\arg(3) = 0$, because positive real numbers make angle 0 with the positive x-axis. Or, from 8.2.7, $\tan\theta = b/a = 0/3 = 0$, so $\theta = \tan^{-1}0 = 0$.

$\arg(-2) = \pi$ (Figure 13), because negative numbers make angle π with the positive x-axis.

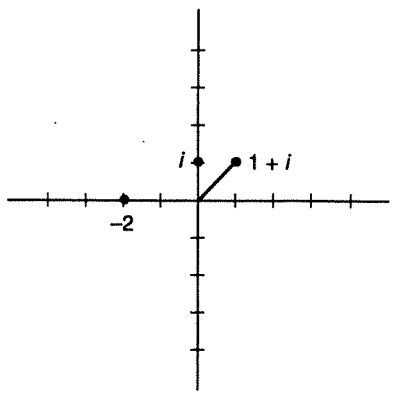


Figure 13: $1 + i$, -2, i. [-5, 5] by [-5, 5].

In the example "arg(-2)" the equation for θ from 8.2.7 ($\tan\theta = 0$) has two important solutions: 0 and π. We want $\theta = \pi$ rather than $\theta = 0$, because $a < 0$. By inspection, $z = -2$ is at angle π from the positive real axis. Real numbers have argument 0 or π.

$\arg(i) = \pi/2$ [second last line of 8.2.7, Figure 13].

$\arg(2 + 3i) = \tan^{-1}(3/2)$

[= .983 radians = 56.31°, Figure 14].

We expressed this answer as "$\tan^{-1}(3/2)$" because it is not a famous angle and the given decimal form for it is not illuminating.

$\arg(-2 + i) = \tan^{-1}(-1/2) + \pi$

$= -.464 + \pi = 2.68$

[= 153.43°, Figure 14].

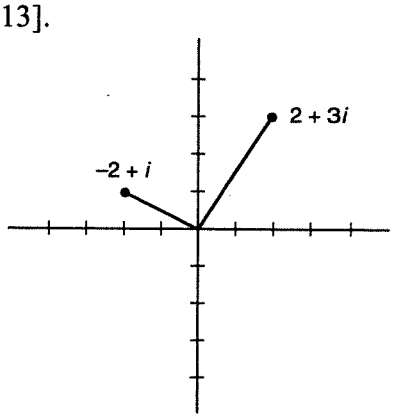


Figure 14: $2 + 3i$, $-2 + i$. [-5, 5] by [-5, 5].

Fortunately, we do not often do examples that are this tricky. For $z = -2 + i$, $\tan\theta = 1/(-2) = -1/2$. But θ is not simply $\tan^{-1}(-1/2)$ because θ is in the second quadrant and $\tan^{-1}(-1/2)$ is in the fourth quadrant (expressed as a negative angle). So adding π gives the second quadrant angle with the same tangent value (recall identity 7.2.10 that says tangent repeats every π radians).

<u>Complex Numbers in Trigonometric Form</u>. Using "$\cos\theta = a/r$" and "$\sin\theta = b/r$" from Figure 12 (or 8.2.7), we can rewrite "$a + bi$."

(8.2.8) $$a + bi = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta).$$

"$r(\cos\theta + i\sin\theta)$" is the trigonometric form (or "polar form") of a complex number $z = a + bi$, where $r = |z|$ as in 8.2.6 and θ is an argument of z as in 8.2.7.

We write "$i \sin\theta$" rather than "$(\sin\theta)i$" to avoid the extra parentheses.

The factor "$\cos\theta + i\sin\theta$" appears a lot in this subject. Since only the "θ" varies in the "$\cos\theta + i\sin\theta$" factor, it is often abbreviated to emphasize the θ. Some authors write "$\cos\theta + i\sin\theta$" as "cis θ," where the three letters c, i, and s abbreviate the phrase "cosine plus i sine." However, this abbreviation is not common in higher mathematics.

Exponential Form. Actually, there is no real need to abbreviate the very common expression "$\cos\theta + i\sin\theta$" at all, because this trigonometric form is both correct and illuminating. Nevertheless, it is often abbreviated because it is a bit long. It turns out, in calculus, that the exponential function with base e that we studied in Chapter 5 yields a shorter and more convenient equivalent expression. We cannot prove it here, but the key identity is "Euler's Formula":

"$e^{i\theta}$" conveniently abbreviates "$\cos\theta + i\sin\theta$." "Exponential form" and "trigonometric form" are two equivalent notations. Everything expressed in the remainder of this section using "exponential form" is nothing more or less than "trigonometric form" rewritten using 8.2.9 and 8.2.11.

(8.2.9) $$e^{i\theta} = \cos\theta + i\sin\theta.$$

This is a complex number on the unit circle in the complex plane, at angle θ from the positive real axis (Figure 15). Since $\cos\pi = -1$ and $\sin\pi = 0$, the famous equation which relates e, i and π follows:

(8.2.10) $$e^{i\pi} = -1.$$

The short exponential form follows immediately from 8.2.8 and 8.2.9:

(8.2.11) $$r\,e^{i\theta} = r(\cos\theta + i\sin\theta).$$

This point is r times as far from the origin and in the same direction as $e^{i\theta}$ on the unit circle (Figure 15). Formula 8.2.11 gives two notations for the same information.

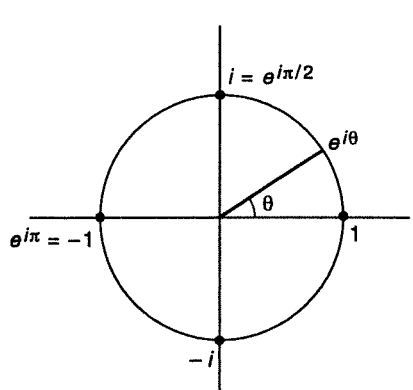


Figure 15: $\cos\theta + i\sin\theta = e^{i\theta}$, on the unit circle in the complex plane.

Example 12: $3 = 3(\cos 0 + i \sin 0) = 3\, e^{0i}$.
The real number 3 is at angle 0 from the positive real axis (Figure 16).

$-4 = 4(\cos \pi + i \sin \pi) = 4\, e^{i\pi}$.
The real number -4 is at angle π from the positive real axis (Figure 16).

$2i = 2[\cos(\pi/2) + i \sin(\pi/2)] = 2\, e^{i\pi/2}$.
The imaginary number $2i$ is at angle $\pi/2$ from the positive real axis (Figure 16).

$1 + i = \sqrt{2}[\cos(\pi/4) + i \sin(\pi/4)] = \sqrt{2}\, e^{i\pi/4}$, because $r = \sqrt{(1^2 + 1^2)} = \sqrt{2}$ and $\tan \theta = 1/1$. Therefore $\theta = \pi/4$, since θ is in the first quadrant (Figure 16).

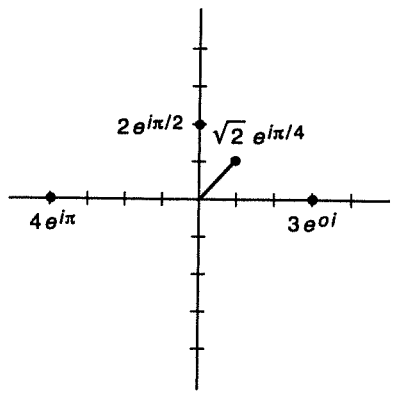


Figure 16: Points expressed in exponential form [-5, 5] by [-5, 5]

Example 13: $1 + i\sqrt{3} = 2[\cos(\pi/3) + i \sin(\pi/3)]$
$= 2\, e^{i\pi/3}$,
because $r = \sqrt{(1^2 + \sqrt{3}^2)} = 2$ and $\tan \theta = \sqrt{3}/1$. Therefore $\theta = \pi/3$, since θ is in the first quadrant (Figure 17).

$2 - 3i = \sqrt{13} \exp[i \tan^{-1}(-3/2)] = \sqrt{13}\, e^{-.98i}$, because $r = \sqrt{(2^2 + 3^2)} = \sqrt{13}$ and $\tan \theta = -3/2$. Therefore, $\theta = \tan^{-1}(-3/2)$, (which is about -.98) since θ is in the fourth quadrant (Figure 17). The notation "$\exp(i\theta)$" instead of "$e^{i\theta}$" in this example allows us to see the tiny exponent more clearly.

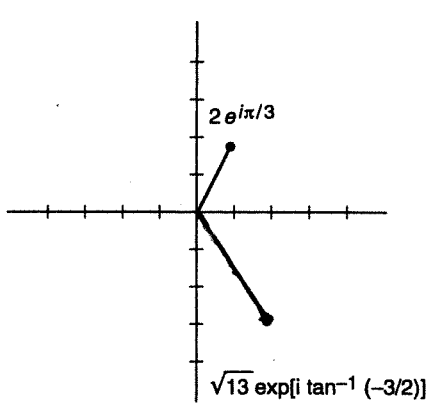


Figure 17: Points expressed in exponential form [-5, 5] by [-5, 5]

Multiplication. Multiplying complex numbers in trigonometric or exponential form yields a remarkable and elegant result:

To multiply complex numbers in trigonometric or exponential form, multiply their moduli and *add* their arguments.

Remember that something similar holds for real numbers: $e^x e^y = e^{x+y}$, which says to multiply exponentials we may add their arguments (5.2.6 or 5.1.1). With complex numbers, addition of arguments is addition of angles. With complex numbers, the angle of a product is the *sum* of the angles of the numbers. The next theorem states this in both trigonometric and exponential notation. It looks long, but it is easily interpreted.

Theorem 8.2.12:
$r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2) = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$.

$$[r_1\exp(i\theta_1)][r_2\exp(i\theta_2)] = r_1r_2\exp[i(\theta_1+\theta_2)].$$

Formula 8.2.12 has two lines which say the same thing in two different notations. The second is shorter and shows that exponential notation is convenient. The theorem explains why multiplying by a complex number produces a rotation. Adding angles is equivalent to rotating about the origin.

Example 9, revisited: Multiplying by i produces a rotation of $\pi/2$ (90°) as we saw in Example 9 (Figure 11). The modulus of i is 1 and its argument is $\pi/2$ (90°). So, $i = 1\exp(i\pi/2)$, according to 8.2.11. Multiplying any complex number by i leaves the modulus unchanged (since $r = 1$) and adds $\pi/2$ to the argument, so the product is the original number rotated by $\pi/2$ (problem B19).

Proof of 8.2.12: This proof relies on regular multiplication of complex numbers in "$a + bi$" form (8.2.3), and then uses the sum-of-angles trigonometric identities for sine and cosine (7.3.1A and B). Note that "trigonometric form" is virtually "$a + bi$" form, but with the "a" and "b" written using trigonometric functions. The first line of the proof expresses the equivalence of the two forms given in 8.2.11.

$$[r_1\exp(i\theta_1)][r_2\exp(i\theta_2)] = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2),$$

Now, factor out the r_1 and the r_2, and multiply out the rest using FOIL (8.2.3).

$$\begin{aligned} = r_1r_2\ \{&(\cos\theta_1)(\cos\theta_2) + i(\cos\theta_1)(\sin\theta_2) \\ &+ i(\sin\theta_1)(\cos\theta_2) + i^2(\sin\theta_1)(\sin\theta_2)\}. \end{aligned}$$

Now, two terms of the sum are real and two are imaginary. Recall $i^2 = -1$. Grouping like terms, this

$$\begin{aligned} = r_1r_2\ \{&(\cos\theta_1)(\cos\theta_2) - (\sin\theta_1)(\sin\theta_2) \\ &+ i[(\cos\theta_1)(\sin\theta_2) + (\sin\theta_1)(\cos\theta_2)]\}. \end{aligned}$$

Look up the two "sum identities" for cosine and sine (7.3.1A and B). Note how they fit perfectly. Therefore, this is

$$= r_1r_2\ \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$$

$$= r_1r_2\exp[i(\theta_1+\theta_2)].$$

The proof is complete.

Aside. It is truly remarkable that "imaginary" numbers should have such a close connection to trigonometry, which began as the study of triangles. Who would have thought that trigonometry and the square root of negative one would be related by such as basic concept as multiplication? Mathematicians have discovered over the ages that abstract concepts that are suggested by mathematics (such as negative numbers and the square root of negative one) often turn out later to be useful and practical in unexpected real-world contexts. Mathematicians use this argument to support "pure" research -- because they know that, no matter how "pure" research may seem at first, there is always a chance that important practical applications will follow over time.

Example 14: Multiply several numbers by the complex number $z = \sqrt{2} + i\sqrt{2}$ and note the graphical effect.

For instance, multiply 3, $1 + 2i$, and $-2 - i$, by that number (P, Q, and S in Figure 18).

Visually, this is easy. The absolute value of the factor z is 2, so the products will be twice as far from the origin. The angle (argument) of z is $\pi/4$ (45°), so the new points will be rotated $\pi/4$. Done. These effects are clearly visible in Figure 18.

To compute the numerical values of the products in rectangular form takes some work. Here are the three results:

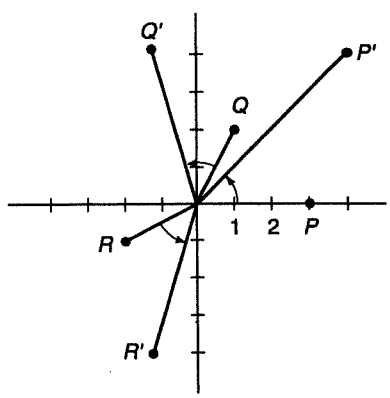


Figure 18: Multiplication by $\sqrt{2} + i\sqrt{2}$. [-5, 5] by [-5, 5].

$Pz = 3(\sqrt{2} + i\sqrt{2}) = 3\sqrt{2} + (3\sqrt{2})i$ [$= P'$, Figure 18].

$Qz = (1 + 2i)(\sqrt{2} + i\sqrt{2}) = -\sqrt{2} + (3\sqrt{2})i$ [$= Q'$, Figure 18].

$Sz = (-2 - i)(\sqrt{2} + i\sqrt{2}) = -\sqrt{2} - (3\sqrt{2})i$ [$= S'$, Figure 18].

Let z be any complex number. Graphically, multiplying a number by z produces a rotation around the origin and an expansion away from the origin or a contraction toward the origin. The rotation is through the angle arg(z). The expansion or contraction is by a factor of $|z|$.

Complex Conjugates. When solving quadratic equations, complex numbers arise in "complex conjugate" pairs.

Definition (8.2.13): The complex conjugate of $a + bi$ is $a - bi$. The complex conjugate of z is often denoted by $\bar{z}$ ("z bar") (Figure 19).

In the Quadratic Formula, the square root term has a "plus or minus" on it:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

If the interior of the square root is negative, the plus sign yields one complex number and the minus sign yields its complex conjugate.

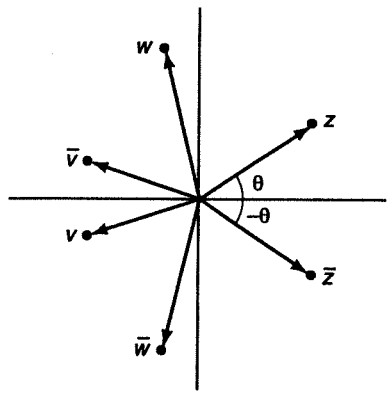


Figure 19: Complex conjugates.

Graphically, the complex conjugate of z is symmetric with z about the real (horizontal) axis. Their real parts are the same and their complex (vertical) parts are opposites (Figure 19).

Example 15: The complex conjugate of "$1 + 2i$" is "$1 - 2i$". Their product is $(1 + 2i)(1 - 2i) = 1 + 4 = 5$, a real number. The cross product term drops out. Also, by 8.2.6, $|1 + 2i| = \sqrt{5}$. So, $z\,\bar{z} = |z|^2$.

The product of a complex number and its conjugate is always a real number, and it is always the square of the modulus (absolute value).

(8.2.14) $\quad z\,\bar{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2$, a real number.

Multiplying it out shows the product is a real number. There is also an illuminating trigonometric explanation. A complex number and its complex conjugate have the same modulus, but opposite angles (Figure 19). So, from 8.2.12, multiplying them together yields the product of the moduli (that is the $|z|^2$ part) and the sum of the angles, which is 0 ($\theta + -\theta = 0$). Angle 0 implies the product is a real number.

Division. So far we have discussed addition, subtraction, and multiplication. Dividing complex numbers by *real* numbers is like multiplication by real numbers; it is done term-by-term, as expected (8.2.5). However, division by a non-real complex number is not quite so easy. In trigonometric form, division is, as expected, the inverse of multiplication, but there is a trick to obtaining the quotient in "$a + bi$" form.

In trigonometric or exponential form, to divide complex numbers, divide the moduli and *subtract* the argument of the denominator from the argument of the

numerator (problem B25). That is, assuming $r_2 \neq 0$,

$$(8.2.15)\quad \frac{r_1 \exp(i\theta_1)}{r_2 \exp(i\theta_2)} = \left(\frac{r_1}{r_2}\right) \exp(i(\theta_1 - \theta_2))$$

Example 16: (Figure 20) $\dfrac{6\exp\left(i\dfrac{\pi}{3}\right)}{2\exp\left(i\dfrac{\pi}{4}\right)}$

$$= 3\exp\left(i\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\right) = 3\exp\left(i\frac{\pi}{12}\right).$$

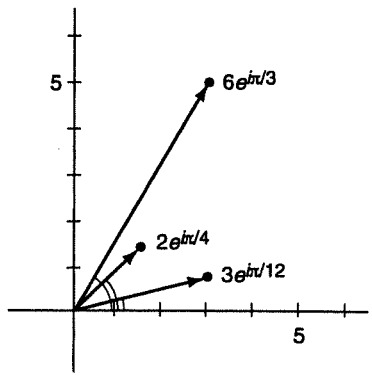


Figure 20: $6\exp(i\pi/3)$, $2\exp(i\pi/4)$, and their quotient.

Division of complex numbers in $a + bi$ form is tricky when the object is to express the result in $a + bi$ form. The key is to multiply both the numerator and denominator by the complex conjugate of the denominator.

Example 17: Write $\dfrac{2+i}{3-2i}$ in $a + bi$ form.

$$\frac{2+i}{3-2i} = \left(\frac{2+i}{3-2i}\right)(1) = \left(\frac{2+i}{3-2i}\right)\left(\frac{3+2i}{3+2i}\right) = \frac{(2+i)(3+2i)}{9+4}$$

Now the division problem has been converted to a straightforward multiplication problem. Use 8.2.3 and 8.2.5 (problem A25).

The point of using the complex conjugate is to make the denominator a real number, so that the remaining division will be by a real number. To finish off, multiply out the top using 8.2.3. In general, the process can be described by the next theorem.

<u>Theorem 8.2.16</u>: $\dfrac{a+bi}{c+di} = \left(\dfrac{a+bi}{c+di}\right)\left(\dfrac{c-di}{c-di}\right)$

$$= \frac{(a+bi)(c-di)}{c^2+d^2}$$

$$= \frac{ac+bd+(bc-ad)i}{c^2+d^2}$$

This is close enough to "$a + bi$" form. The pattern is too complicated to remember in this abstract form. Instead, simply remember to multiply both the top and bottom by the complex conjugate of the bottom.

Example 18: Evaluate, in $a + bi$ form, $\dfrac{1+i}{1-i}$.

$$\frac{1+i}{1-i} = \left(\frac{1+i}{1-i}\right)\left(\frac{1+i}{1+i}\right) = \frac{0+2i}{1^2+1^2} = i.$$

Problem B20 asks for a trigonometric, visual, explanation of why this quotient reduces to i.

Conclusion. Complex numbers are expressed in three common forms, "$a + bi$" form, trigonometric (polar) form, and exponential form. Trigonometric and exponential forms are especially valuable for multiplication and division, but are not convenient for addition or subtraction. Multiplication of complex numbers is closely related to rotation in two dimensions.

Terms: complex number, $a + bi$ form, real part, imaginary part, complex plane, modulus, argument, trigonometric form, exponential form, complex conjugate.

Exercises for Section 8.2, "Complex Numbers":

A1.* Given two complex numbers in "$a + bi$" form, how can you obtain their sum?

^^^^ Sketch the location in the complex plane of

A2. a) $P = 2 + i$ b) $Q = -1 + 2i$ c) $R = 3i$
A3. a) $P = 2 - i$ b) $Q = -2i$ c) $R = -1 - i$
A4. a) $P = \cos \pi + i \sin \pi$. b) $Q = \cos(\pi/3) + i \sin(\pi/3)$
A5. a) $P = \cos(-\pi/2) + i \sin(-\pi/2)$ b) $Q = \cos(3\pi/4) + i \sin(3\pi/4)$
A6. a) $P = e^{i\pi/6}$ b) $Q = e^{i\pi}$.
A7. a) $P = e^{i\pi/2}$ b) $Q = e^{i\pi/4}$.
A8. a) $P = 3e^{i\pi/3}$ b) $Q = 2e^{i5\pi/4}$.
A9. a) $P = 3[\cos(\pi/6) + i \sin(\pi/6)]$. b) $Q = 2[\cos(3\pi/4) + i \sin(3\pi/4)]$
A10. a) $P = 4[\cos(5\pi/6) + i \sin(5\pi/6)]$. b) $Q = 3[\cos(-\pi/4) + i \sin(-\pi/4)]$

^^^^ Simplify:

A11. a) $4 + i + 5 + 7i$ b) $4 + i - (5 + 7i)$
A12. a) $(4 + i)(3 + 2i)$. b) $(4 + i)/(3 - 2i)$.
A13. a) $(2 + 3i)(5 - i)$. b) $(2 + 3i)/(5 + i)$.
A14. a) $(1 + 3i)i$ b) $(1 + 2i)/i$.

^^^^ Write the given complex number in exponential form.

A15. a) i b) -1 c) $2 + 2i$ d) $1 + 4i$ e) $-1 + i\sqrt{3}$.
A16. a) $-i$ b) 1 c) $1 + i\sqrt{3}$ d) $-1 + i$. e) $2 + 3i$.

A17. Write the complex numbers in A15 in trigonometric form.
A18. Write the complex numbers in A16 in trigonometric form.

^^^^ Convert these to "$a + bi$" form.

A19. a) $3 \exp(i\pi/6)$ b) $4 \exp(i3\pi/4)$. A20. a) $5 \exp(\pi i/4)$. b) $2 \exp(i2\pi/3)$.

^^^^ Simplify:

A21. $[2\exp(i\pi/4)][6\exp(i3\pi/4)]$ A22. $[5\exp(i\pi/12)][2\exp(i\pi/6)]$

A23. Divide: $\dfrac{20\exp(3\pi i/4)}{4\exp(\pi i/3)}$ A24. Divide: $\dfrac{3\exp(\pi i/2)}{60\exp(\pi i/6)}$

#A25. Simplify the expression from Example 17: $(2 + i)(3 + 2i)/(9 + 4)$.

A26. Draw and label a picture of the complex plane to illustrate "$e^{i\pi} = -1$."

A27. What is the argument (angle) of negative real numbers in the complex plane? Explain how multiplying a real number by a negative real number fits the idea of rotation in 8.2.12.

^^^^^^^^

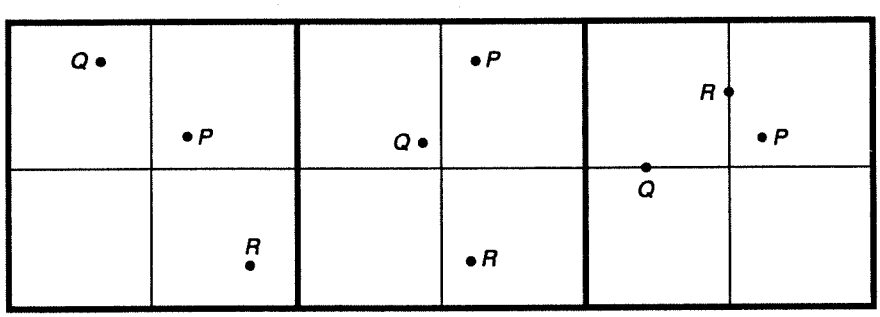
B1.* Sketch a picture that illustrates how to convert back and forth between "$a + bi$" form and trigonometric or exponential form.

B2.* a) State the two most important facts for converting a number in "$a + bi$" form to a number in trigonometric or exponential form.
b) State the two most important facts for converting a number in trigonometric or exponential form to a number in "$a + bi$" form.

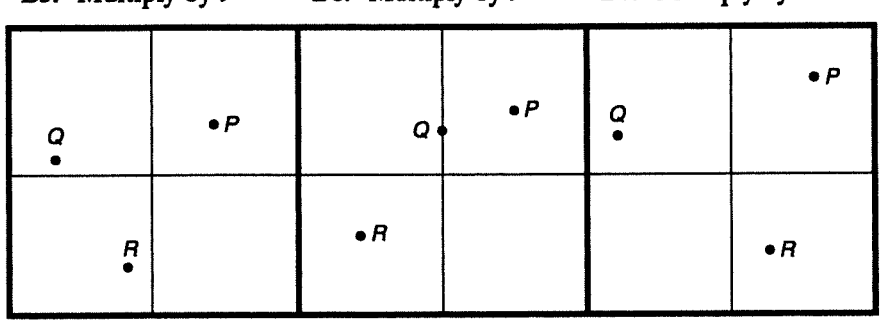
B3.* Given two complex numbers in "$a + bi$" form, how can you obtain their product?

B4. Exponential and trigonometric forms are very similar. Are there any essential differences?

^^^^ The pictures locate P, Q, and R in the complex plane. Reproduce the picture and find the location of the resulting point when each is multiplied or divided by the indicated complex number.



B5. Multiply by i **B6. Multiply by i** **B7. Multiply by $1 + i$.**



B8. Divide by i **B9. Divide by $2i$** **B10. Divide by $1 + i$**

B11.* a) Thinking only of real numbers, what is the graphical effect on the number line of the function, "multiply by c"?
b) In the complex plane, what is the graphical effect of the function "multiply by w"?

B12. Using the points in problem B5, illustrate the result of "add i" to each. Assume the window is [-5, 5] by [-5, 5].

B13. Using the points in problem B7, illustrate the result of "add $1 - i$" to each. Assume the window is [-5, 5] by [-5, 5].

B14.* What is the graphical effect of the function "add w"?

B15.* Given two complex numbers in "$a + bi$" form, how can you obtain their quotient?

B16.* Given two complex numbers in exponential form, how can you obtain their product?

B17.* Give the "trigonometric form" explanation of why $i^2 = -1$.

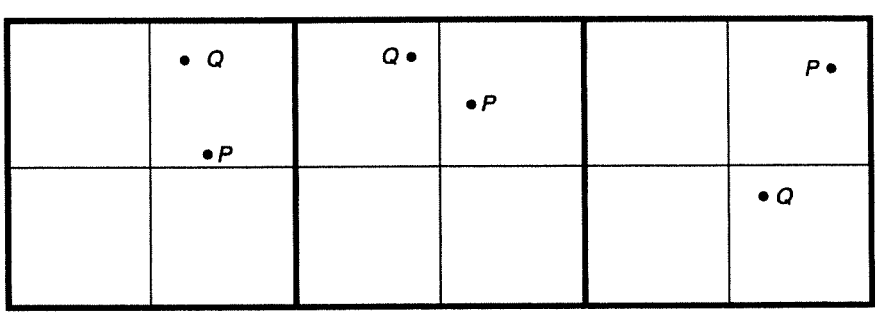
B18. The modulus function has properties on the complex numbers that the absolute value function has on the real numbers. So it can reasonably be called the "absolute value" function on the complex numbers. Prove a) The modulus of a real number is its absolute value.
b) $|z| \geq 0$.

B19. Use abstract notation and 8.2.12 to show that multiplying z by i produces a rotation of $\pi/2$, as illustrated in Example 9 (Figure 11) and as argued below 8.2.12.

B20. Explain, trigonometrically and visually, why the quotient in Example 18 is simply i.

B21. The Quadratic Formula gives two solutions to a quadratic equation. Express their product.

[The remaining problems are challenging.]
^^^^^^^^ The pictures below illustrate P and Q on a square scale, where $Q = wP$. Estimate the solution for the unknown complex number w. (You do not need to know the scale!)



B22. B23. B24.

B25. Prove 8.2.15 on division of complex numbers in trigonometric form.

B26. Find $\sqrt{i}$ in "$a + bi$" form.

B27. [Part (b) is a famous theorem about complex numbers known as DeMoivre's Theorem.]
a) Use trigonometric form to express z^2 in terms of the modulus and argument of z.
b) Use exponential or trigonometric form to express z^n in terms of the modulus and argument of z.
c) Recall that $1 = e^{2k\pi i}$, for any integer k. Use 3 different values of k to obtain 3 distinct solutions to "$z^3 = 1$." d) Plot them in the complex plane.

B28. a) Use the ideas in B27 to find n "n^{th} roots of unity" by solving "$z^n = 1$."
b) If all n n^{th} roots are plotted in the complex plane a pattern results. What pattern?

B29. Fractals. Fractal images are increasingly common because fast computers are able to do the immense amount of computation required to create the pictures. Here is the idea behind the particular fractal in Figure 1. It uses repeated composition with the same simple function, $f(z) = z^2 - 1$. Consider all possible complex numbers, z, one by one. Pick one, apply f to it, then apply f again to the image, and then again and again. So the initial point is repeatedly moved in the complex plane. Now consider the question, "Does it move further and further from the origin, (move off to "infinity") or does its orbit (sequence of images) stay in a bounded region near the origin?" Theoretically, we could do this for every point in the plane and keep track of the points that move off to infinity. There will be a region of such points, and the "Julia set" is the boundary of that region, which is highly irregular for most functions, f, even when f appears to be simple. For example, Figure 1 graphs the Julia set of the very simple f given by $f(z) = z^2 - 1$. Here is the problem: Find all the points in the complex plane ("fixed points") that have the same image as argument when this function is applied.

Section 8.3. Polar Coordinates

Polar coordinates are a way to locate points in the plane. Rectangular coordinates use two numbers to locate positions in the plane, one for the horizontal position and one for the vertical position. Polar coordinates also use two numbers, one for the distance from the origin and the other for the angle with the positive x-axis.

Fix a point for the origin, O (Figure 1). The origin is the pole of "polar coordinates." Fix a ray (half line, OA, like a positive x-axis) from the pole and label it with a scale. To locate the point P in this "polar coordinate system," give the ordered pair (r, θ), where r is a directed (positive or negative) distance and θ is an angle measured counterclockwise as illustrated in Figure 1.

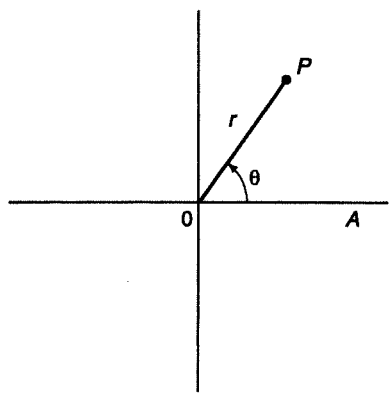


Figure 1: Polar coordinates. P is the point (r, θ).

It is possible to associate many different angles with any point P. Figure 2 illustrates $(3, \pi/6)$. Figure 3 labels the same point with a negative angle. The angles are coterminal (terminate in the same place).

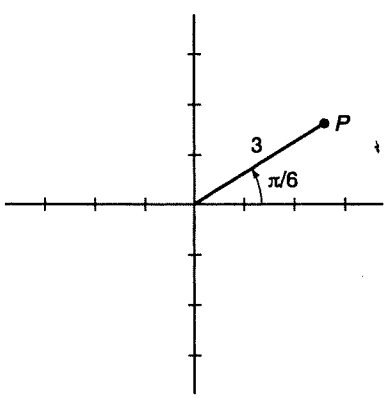


Figure 2: $P = (3, \pi/6)$ [-4, 4] by [-4, 4].

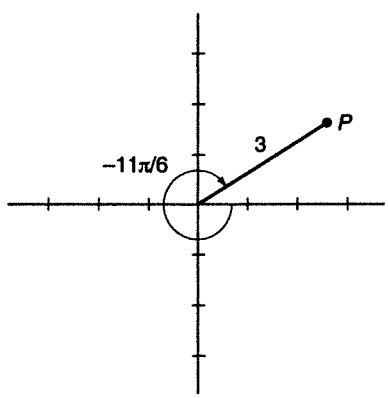


Figure 3: $P = (3, -11\pi/6)$ [-4, 4] by [-4, 4].

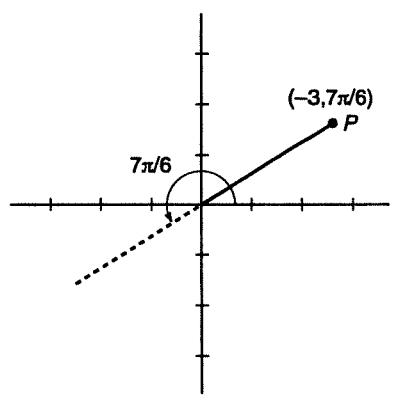


Figure 4: $(-3, 7\pi/6)$ [-4, 4] by [-4, 4].

Usually we think of "r" as representing distance from the origin, but it is actually "directed" distance because r can be negative. Figure 4 illustrates the same point as Figures 2 and 3, but with the angle in the opposite direction and with r negative. Figures 2, 3 and 4 show that one major difference between rectangular coordinates

and polar coordinates is that points do not have a unique representation in polar coordinates.

In rectangular coordinates the grid is determined by setting x equal to a constant (for the vertical grid lines) and y equal to a constant (for the horizontal grid lines). In polar coordinates the grid is determined by setting r equal to a constant r_0 (to obtain circles with radius r_0 centered at the origin) and setting θ equal to a constant θ_0 (to obtain lines through the origin at angle θ_0 with the positive x-axis, Figure 5).

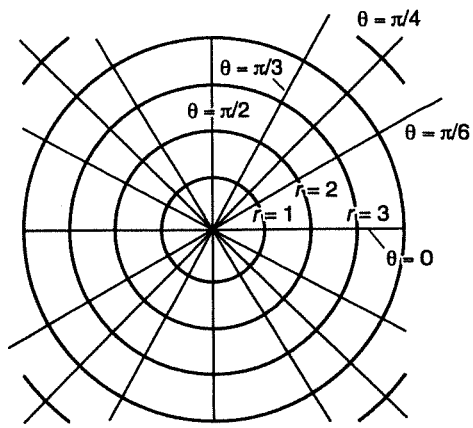


Figure 5: The polar coordinate grid.

Example 1: Graph "$r = 2$" in polar coordinates.

Since θ is not mentioned, θ can be anything. Figure 6 graphs all points 2 units from the origin. The graph is a circle.

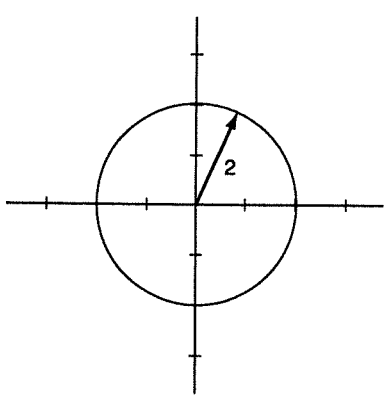


Figure 6: $r = 2$.
[-4, 4] by [-4, 4].

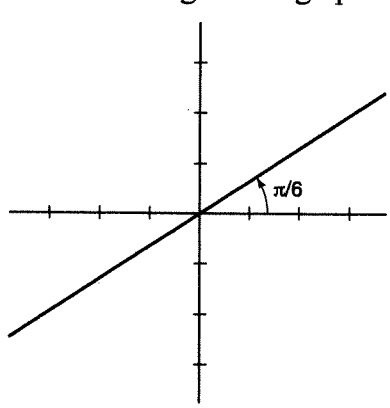


Figure 7: $\theta = \pi/6$.
[-4, 4] by [-4, 4].

Example 2; Graph "$\theta = \pi/6$" in polar coordinates.

Since r is not mentioned, r can be anything, including negative values. Figure 7 graphs all points at angle $\pi/6$ with the positive x-axis.

Conversion Between Coordinate Systems. The relationship between polar and rectangular representations follows easily from Figure 8. Conversion from polar to rectangular coordinates yields a unique rectangular coordinate representation:

(8.3.1) $x = r\cos\theta$ and $y = r\sin\theta$.

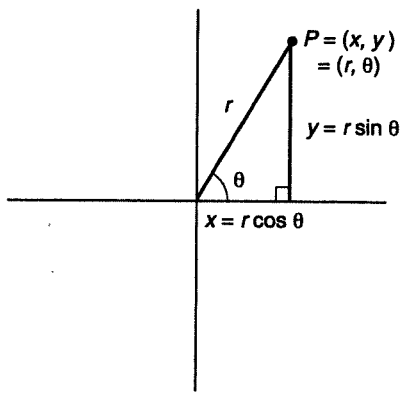


Figure 8: Polar and rectangular representations of P.

Converting from rectangular to polar coordinates does not yield a unique polar coordinate representation. Given the rectangular representation (x, y), r and θ satisfy

(8.3.2) $r^2 = x^2 + y^2$ and $\tan\theta = y/x$.

These equations yield two solutions for r (positive or negative) and an infinite number of solutions for θ, including two in $[0, 2\pi)$. This means we must make some choices. Usually, but not always, we chose the positive value of r. Then we choose θ to match the quadrant of (x, y).

Example 3: Give the polar-coordinate representation of the rectangular-coordinate point (1, 1) (Figure 9).

From 8.3.2,

$r^2 = 1^2 + 1^2 = 2$ and $\tan\theta = 1/1 = 1$.

Now, make some choices. Our preference for positive r and first quadrant angles leads us to prefer $r = \sqrt{2}$ and $\theta = \tan^{-1} 1 = \pi/4$. So one polar representation of the point is $(\sqrt{2}, \pi/4)$.

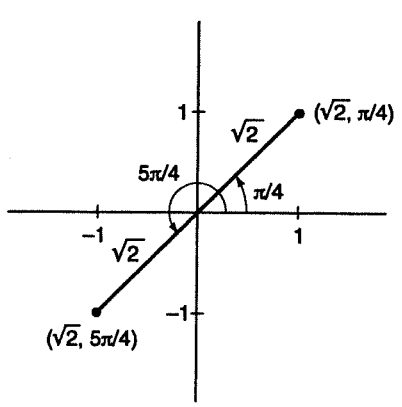


Figure 9: (1, 1), (-1, -1) and their polar-coordinate representations. [-2, 2] by [-2, 2]

Example 4: Give the polar-coordinate representation of the rectangular-coordinate point (-1, -1) (Figure 9).

$r^2 = (-1)^2 + (-1)^2 = 2$ and $\tan\theta = (-1)/(-1) = 1$.

These are the *same equations* for r and θ as in Example 3, but the point is different. Clearly we must choose a different solution because the point is in the third quadrant.

Again, choose $r = \sqrt{2}$. Then we must choose a third quadrant angle θ with $\tan\theta = 1$. Tangent repeats every π radians, so $\tan(\pi/4+\pi)$ is also equal to 1. A polar-coordinate representation is $(\sqrt{2}, \pi/4+\pi) = (\sqrt{2}, 5\pi/4)$. Another is $(-\sqrt{2}, \pi/4)$.

To convert from rectangular coordinates, if we choose $r > 0$, a polar-coordinate representation of (x, y) is

$$\left(\sqrt{x^2 + y^2}, \tan^{-1}\left(\frac{y}{x}\right)\right) \text{ if } (x, y) \text{ is in the first or fourth quadrant, and}$$

(8.3.3)

$$\left(\sqrt{x^2 + y^2}, \tan^{-1}\left(\frac{y}{x}\right) + \pi\right) \text{ if } (x, y) \text{ is in the second or third quadrant.}$$

Because angles that differ by 2π, or any multiple of 2π, are coterminal, any θ may be replaced by $\theta \pm 2n\pi$. That is, in polar coordinates,

(8.3.4) (r, θ) and $(r, \theta + 2n\pi)$ represent the same point.

Instead of memorizing these formulas, learn Figure 8, which contains the information of all the formulas in this section.

Also, since angles that differ by π are opposite, the same point is obtained by also using the opposite directed distance.

(8.3.5) (r, θ) and $(-r, \theta + \pi)$ represent the same point.

Example 5: Give several alternative polar-coordinate representations of $(5, 2\pi/3)$.

One alternative is to add 2π to the angle: $(5, 2\pi/3) = (5, 2\pi/3+2\pi) = (5, 8\pi/3)$. Another alternative is to subtract 2π: $(5, 2\pi/3) = (5, 2\pi/3 - 2\pi) = (5, -4\pi/3)$. A third alternative is to use a negative r as in 8.3.5: $(5, 2\pi/3) = (-5, 2\pi/3 + \pi) = (-5, 5\pi/3)$.

Conversion from polar-coordinates to rectangular coordinates is not tricky.

Example 6: Give the rectangular-coordinate representation of the polar-coordinate point $(2, \pi/6)$ (Figure 2).

From 8.3.1, $x = r\cos\theta = 2\cos(\pi/6) = 2(\sqrt{3}/2) = \sqrt{3}$. $y = r\sin\theta = 2(1/2) = 1$. The unique rectangular-coordinate representation is $(\sqrt{3}, 1)$.

<u>Well-known Polar-Coordinate Graphs</u>. Most graphs are given in rectangular coordinates, but there are occasionally reasons to use polar coordinates. Section 8.3 on complex numbers has an application to multiplying complex numbers. Polar-coordinate graphs are most appropriate in applications where the distance to a particular point is key. For example, when studying the effect of the gravitational pull of the sun on the planets, the distance of the planets to the sun is important. By locating the origin at the sun, that distance becomes r and the position in orbit around

the sun is described by (r, θ). We will study this polar representation of orbits in Section 9.3 on conic sections.

To express polar-coordinate functions, r is usually treated as a function of θ (rather than θ as a function of r). Here are some graphs that are easy to express in polar coordinates.

Example 7: Graph $r = \cos\theta$ in polar coordinates.

The graph is the circle in Figure 10. We can show it is a circle by converting to rectangular coordinates.

Multiply by r to obtain

$$r^2 = r\cos\theta.$$

By 8.3.2 and 8.3.1, this is

$$x^2 + y^2 = x.$$
$$x^2 - x + y^2 = 0.$$
$$x^2 - x + 1/4 + y^2 = 1/4.$$
$$(x - 1/2)^2 + y^2 = (1/2)^2.$$

This is the standard form (3.1.13) of a circle centered at (1/2, 0) with radius 1/2.

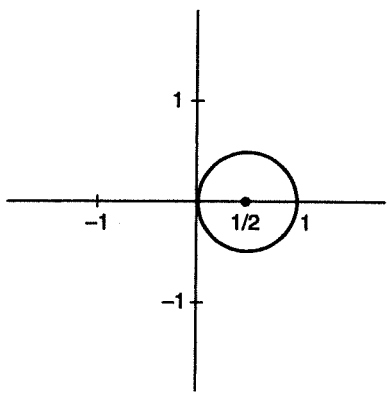


Figure 10: $r = \cos\theta$. [-2, 2] by [-2, 2]. $0 \le \theta < 2\pi$ or $0 \le \theta < \pi$.

Since $\cos\theta$ repeats every 2π, there is no need to use a domain larger than $[0, 2\pi)$ (one complete revolution). Actually, this particular graph is complete after only half a revolution. The first quadrant angles yield the top half. In the second quadrant cosine is negative, so $r = \cos\theta$ is negative. Therefore, rather than yielding points in the second quadrant, second quadrant angles yield points in the opposite quadrant, the fourth quadrant. These form the bottom half of the circle. Then, when θ is a third quadrant angle, $\cos\theta$ is still negative and the points in the first quadrant are retraced. Finally, when θ is a fourth quadrant angle, $\cos\theta$ is positive and the fourth quadrant points are retraced.

Calculator Exercise 1: Watch the graph of $r = \cos\theta$ develop on a graphics calculator using domain $[0, 2\pi)$. Which points are created first? Which are created last?

If your calculator does not have a "polar" coordinate graphing mode, you may use "parametric" mode instead. To graph "$r = f(\theta)$" parametrically, simply let $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$.

For example, to graph "$r = \cos\theta$" parametrically, let $x = (\cos\theta)^2$ [this is $r\cos\theta$] and $y = (\cos\theta)(\sin\theta)$ [this is $r\sin\theta$].

Example 8: Graph $r = 1 + \sin\theta$.

See Figure 11. This shape is called a "cardioid" from the Greek "kardia" meaning "heart." It is not convenient to express in rectangular coordinates.

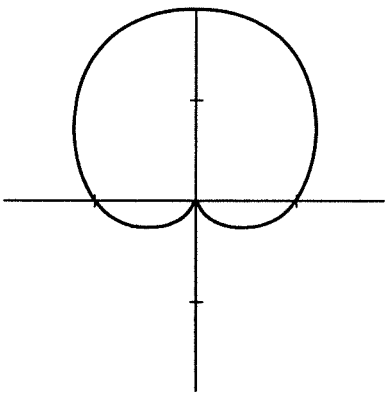


Figure 11: $r = 1 + \sin\theta$.
[-2, 2] by [-2, 2].
$0 \le \theta < 2\pi$.

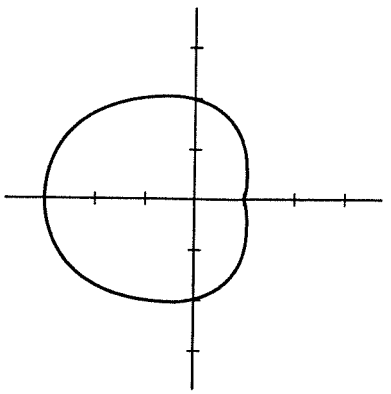


Figure 12: $r = 2 - \cos\theta$.
[-4, 4] by [-4, 4].
$0 \le \theta < 2\pi$.

Example 9: Graph $r = 2 - \cos\theta$.

In contrast to Example 8 ($r = 1 + \sin\theta$), here r is never zero. This shape is called a "limacon," as are other shapes with similar equations (Problems B17-20. Some even have two loops, one inside other (problems B18 and B19).

Example 10: The simple equation "$r = \theta$" forms the "Spiral of Archimedes." In it the distance from the origin equals the angle, so as the angle increases the curve spirals out (Figure 13) (problem B42).

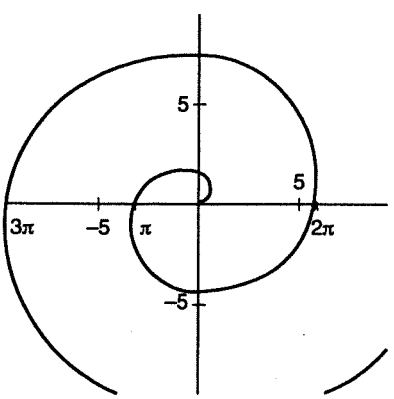


Figure 13: A Spiral of Archimedes. $r = \theta$.
[-10, 10] by [-10, 10].
$0 \le \theta$. Points for $\theta > 12$ are off the window.

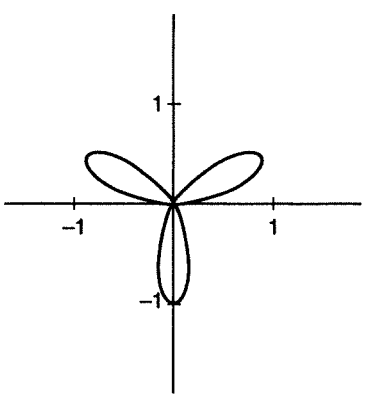


Figure 14: $r = \sin(3\theta)$.
[-2, 2] by [-2, 2].
$0 \le \theta < 2\pi$.

Example 11: Various leaf shapes can be described by "$r = \sin(n\theta)$" for various integer values of n. Figure 14 illustrates "$r = \sin(3\theta)$."

Calculator Exercise 2: Watch the graph of "$r = \sin(3\theta)$" develop on a graphics calculator. Which leaf develops second? (problem A33).

<u>**Symmetry**</u>. Symmetry about the x- or y-axis is often visible in polar-coordinate graphs (Figures 10, 11, and 12). "$\cos\theta$" is symmetric about $\theta = 0$, which is the x-axis, so equations expressed in terms of "$\cos\theta$" have graphs that are symmetric about the x-axis (Figures 10 and 12, see also problem B2). "$\sin\theta$" is symmetric about $\theta = \pi/2$, which is the y-axis, so equations expressed in terms of "$\sin\theta$" have graphs that are symmetric about the y-axis (Figure 11, see also problem B3).

Example 12: Figure 15 exhibits the graph of $r = 1 + \sin(2\theta)$. It uses the sine function, but its graph is not symmetric about the y-axis. Why not?

The function "$\sin\theta$" is symmetric about $\pi/2$. But the argument of sine in the expression "$1 + \sin(2\theta)$" is not θ, it is 2θ. So symmetry occurs about θ such that $2\theta = \pi/2$, that is, about $\theta = \pi/4$ (45°, the line $y = x$). In Figure 15 you can see the symmetry about the diagonal line $\theta = \pi/4$.

Another type of symmetry in Figure 15 is point symmetry about the origin -- for every point there is another equidistant on the opposite side of the origin. That is, when (r, θ) is on the graph, so it $(r, \theta + \pi)$. This always happens when r is a function of sine or cosine of 2θ, as in Figure 15 (Problem B4).

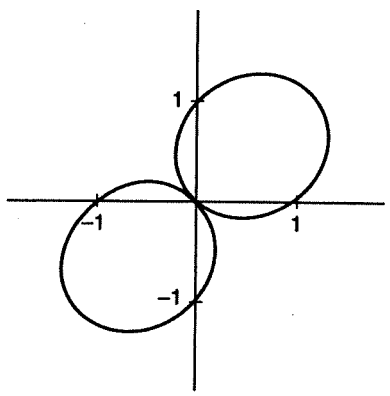


Figure 15: $r = 1 + \sin(2\theta)$. [-2, 2] by [-2, 2]. $0 \le \theta < 2\pi$.

There is another simple cause of point symmetry. The two points (r, θ) and $(-r, \theta)$ are point symmetric about the origin. Therefore, for any $f(\theta)$, the graph of "$r^2 = f(\theta)$" will be point symmetric about the origin because the negative of any r that solves it will also solve it (problems A37, A38, and B27).

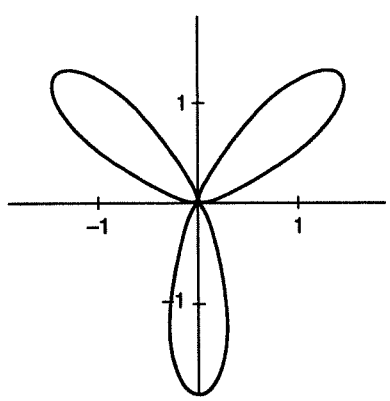


Figure 16: $r = 2\sin(3\theta)$. [-2, 2] by [-2, 2]. [Compare with Figure 14.]

<u>**Scale**</u>: The graphs of "$r = f(\theta)$" and "$r = cf(\theta)$" differ by the scale factor c. The second graph is c times as large, expanded *radially*.

Example 11, modified: Figure 14 gives the graph of "$r = \sin(3\theta)$" from Example 11. Figure 16 shows the graph of "$r = 2\sin(3\theta)$" would be twice the size, expanded away from the origin.

Conclusion. The relationship between rectangular and polar coordinates is illustrated in Figure 8: $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, and $\tan\theta = y/x$. Polar-coordinate representations of points are not unique.

Exercises for Section 8.3, "Polar Coordinates":

^^^^ The point is given in rectangular coordinates. Convert it to polar coordinates with $r \geq 0$ and $0 \leq \theta < 2\pi$.

A1. (2, 2) A2. (-2, 0) A3. (0, 4) A4. (0, -3)
A5. (3, 0) A6. (-2, 2) A7. (1, $\sqrt{3}$) A8. ($\sqrt{3}$, 1)

^^^^ The point is given in polar coordinates, using radians. Convert it to rectangular coordinates.

A9. (3, $\pi/2$) A10. (4, $-\pi/2$) A11. (2, 0) A12. (5, π)
A13. (4, $\pi/4$) A14. (6, $\pi/3$) A15. (2, $\pi/6$) A16. (-2, $5\pi/6$)

^^^^ The point is given in polar coordinates. Give three alternative polar-coordinate representations of the point.

A17. (2, π) A18. (0, $\pi/2$) A19. (5, 0) A20. (-4, 0)

^^^^ Sketch the graph.

A21. $r = 5$. A22. $r = 4$ A23. $\theta = \pi/12$ A24. $\theta = -\pi/4$

^^^^ Use the cited figure to sketch the graph

A25. $r = 3\cos\theta$ (Figure 10) A26. $r = 2 + 2\sin\theta$ (Figure 11)
A27. $r = 1 - (\cos\theta)/2$ (Figure 12) A28. $r = 5\sin(3\theta)$ (Figure 14)

^^^^ Determine the length of one leaf.

A29. $r = 4\sin(3\theta)$ A30. $r = 7\cos(2\theta)$

A31. Compare the graphs of "$r = f(\theta)$" and "$r = 5f(\theta)$."

A32. Watch the graph of $r = \sin(2\theta)$ develop on a graphics calculator using domain $[0, 2\pi)$. a) The points that appear in the fourth quadrant are associated with angles in which quadrant? b) How can that be?

A33. Do Calculator Exercise 2: Watch the graph of $r = \sin(3\theta)$ develop on a graphics calculator using domain $[0, 2\pi)$. a) Which leaf develops second? b) Which values of θ yield points in the second leaf? c) The points that appear in the third quadrant are associated with angles in which quadrant? d) How can that be?

A34. "$r = 1 + \sin\theta$" can be regarded as "$r = f(\sin\theta)$." Give f.

A35. "$r = 2 - \cos\theta$" can be regarded as "$r = f(\cos\theta)$." Give f.

A36. "$r = \sin\theta$" can be regarded as "$r = f(\sin\theta)$." Give f.

A37. a) Graph "$r^2 = \theta$, $0 \leq \theta \leq 2\pi$." b) What symmetry does it have?

A38. a) Graph "$r^2 = \cos\theta$." b) Give all the symmetries it has.

^^^^^^^^

B1.* Sketch a figure that exhibits how to convert between rectangular and polar coordinates.

B2. a) Draw a picture to illustrate why (r, θ) and $(r, -\theta)$ are symmetric about the x-axis.
b) Draw a unit-circle rectangular-coordinate picture to illustrate why $\cos(-\theta) = \cos\theta$.
c) If $r = f(\cos\theta)$ for some f (as in Figures 10 and 12), and (r, θ) is on the graph, what other point is automatically on the graph? Why? d) Sketch a figure of your choice (your choice of f) to illustrate the resulting symmetry about the x-axis.

B3. a) Draw a picture to illustrate why $(r, \pi/2 + \theta)$ and $(r, \pi/2 - \theta)$ are symmetric about the y-axis.
b) Draw a unit-circle rectangular-coordinate picture to illustrate why $\sin(\pi/2 + \theta) = \sin(\pi/2 - \theta)$. c) If $r = f(\sin\theta)$ for some f (as in Figure 11), and $(r, \pi/2 - \theta)$ is on the graph, what other point is automatically on the graph? Why? d) Sketch a figure of your choice (your choice of f) to illustrate the resulting symmetry about the y-axis.

B4. a) Draw a picture to illustrate why (r, θ) and $(r, \theta+\pi)$ are point symmetric about the origin. b) If $r = f(\sin(2\theta))$ for some f and (r, θ) is on the graph, why is $(r, \theta+\pi)$ automatically on the graph? [Hint: "$\sin(\alpha+2\pi) = \sin\alpha$" is the relevant trig identity.]
c) Sketch a figure of your choice (your choice of f) to illustrate the resulting point symmetry about the origin.

^^^^ The figure locates $P = (r, \theta)$. Sketch the figure and on it locate
B5. a) $A = (-r, \theta)$. b) $B = (r, -\theta)$. c) $C = (r, \theta+\pi)$ d) $D = (r, \theta+\pi/2)$
B6. a) $A = (2r, \theta)$. b) $B = (r, \theta - \pi/2)$ c) $C = (r, -\theta)$ d) $D = (-r, \theta)$

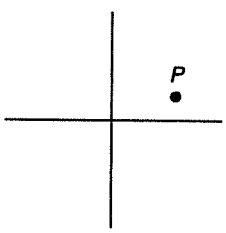


Figure for B5

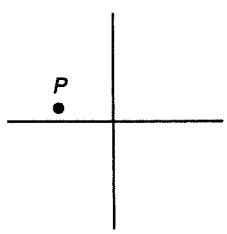


Figure for B6

B7. Express "$x = c$" in polar coordinates. B8. Express "$y = k$" in polar coordinates.

^^^^ Sketch the graph of
B9. $r = \sin\theta$ B10. $r = 1 + \cos\theta$ B11. $r = 2 - \sin\theta$ B12. $r = \cos(3\theta)$

^^^^ (Following B9-B12) Each of the graphs in B9-B12 is the same shape as a graph in an example in this section, but rotated about the origin. For each give the figure number of the similar graph and give the rotation from the text graph to the new graph.
B13. Problem B9. B14. Problem B10. B15. Problem B11. B16. Problem B12.

^^^^ Sketch the graph.
B17. $r = 3 - 2\sin\theta$. B18. $r = 2 - 3\sin\theta$.
B19. $r = 2 - 3\cos\theta$. B20. $r = 2 + \cos\theta$.
B21. $r = 3\sin(2\theta)$. B22. $r = 4\cos(2\theta)$.
B23. $r = 4\sin(4\theta)$. B24. $r = 5\cos(4\theta)$.
B25. $r^2 = 5\cos(2\theta)$ B26. $r = 7\cos(3\theta)$.

B27. a) Draw a picture to illustrate why (r, θ) and $(-r, \theta)$ are point symmetric about the origin. b) If $r^2 = f(\theta)$ and (r, θ) is on the graph, what other point is automatically on the graph? Why? c) Sketch a figure of your choice to illustrate the resulting point symmetry about the origin.

B28. The graph of "$r = f(\sin(2\theta))$" is symmetric about $\theta = \pi/4$ for any f. a) Why?
b) Give an f (your choice) and sketch the graph, including the line of symmetry.

B29. The graph of "$r = f(\cos(2\theta))$" is symmetric about the x-axis for any f. a) Why?
b) Give an f (your choice) and sketch the graph, including the line of symmetry.

B30. Suppose we wish to graph "$r = f(\sin(3\theta)$" for some f. There will be a line of symmetry of the graph in the first quadrant. a) Use trig to show which line it is.
b) Sketch "$r = \sin(3\theta)$" and note the line of symmetry on the graph (To extend this result, see also Problem B41).

B31. Suppose we wish to graph "$r = f(\cos(3\theta))$" for some f. There will be a line of symmetry of the graph in the first quadrant (not just $\theta = 0$). a) Use trig to show which line it is. b) Sketch "$r = \cos(3\theta)$" and note the line of symmetry on the graph (To extend this result, see also Problem B40).

B32. Look at graphs of $r = \sin(2\theta)$, $r = \sin(3\theta)$, and $r = \sin(4\theta)$.
a) Guess how many leaves the graph of $r = \sin(n\theta)$ has for integer values of n.
b) Look at a graph of $r = \sin(5\theta)$ to check your guess. Was your guess right for $n = 5$?

B33. Let L be the line $x = -2$. Let P be a point to the right of line L. a) Find a polar-coordinate equation for the set of all points P such that the distance from P to the line L is equal to the distance from P to the origin. b) Solve for r. c) What shape is the curve?

B34. Let L be the line $x = -3$. Let P be a point to the right of line L. Find a polar-coordinate equation for the set of all points P such that the distance from P to the line L is twice the distance from P to the origin. Solve for r.

B35. Let L be the line $x = -3$. Let P be a point to the right of line L. Find a polar-coordinate equation for the set of all points P such that the distance from P to the line L is half the distance from P to the origin. Solve for r.

B36. Recall that the rectangular-coordinate graphs of "$\sin x$" and "$\cos x$" have the same shape. Either can be expressed as the other shifted left or right. Now consider the graph of any equation "$r = f(\cos\theta)$" for some f (for instance, Figure 10, $r = \cos\theta$ and Figure 12, $r = 2 - \cos\theta$). a) What does the graph of "$r = f(\sin\theta)$" [cosine is replaced by sine] look like compared to the old one? State this result clearly. b) State the relevant trig identity. c) Explain why the trig identity justifies your result in part (a).

B37. Consider any graph with equation "$r = f(\sin\theta)$" for some f (for instance, Figure 11, $r = 1 + \sin\theta$). a) What does the graph of "$r = f(\cos\theta)$" [sine is replaced by cosine] look like compared to the old one? State this result clearly. b) State the relevant trig identity.
c) Explain why the trig identity justifies your result in part (a).

B38. Suppose a graph is symmetric about both the x- and y-axes. Prove that it must be point symmetric about the origin.

B39. Suppose the unit square with sides connecting (0, 0), (1, 0), (1, 1), and (0, 1) [rectangular coordinates] were expressed in polar coordinates by "$r = f(\theta)$." Would the graph of "$r = 2f(\theta)$" be the graph of a square, or would the shape change? Indicate why or why not.

B40. Suppose we wish to graph "$r = f(\cos(3\theta))$." For any f there will be lines of symmetry of the graph because of the symmetry of cosine. a) Find all those the lines of symmetry.
b) Sketch "$r = \cos(3\theta)$" and note the lines of symmetry on the graph.

B41. Suppose we wish to graph "$r = f(\sin(3\theta))$." For any f there will be lines of symmetry of the graph because of the symmetry of sine. a) Find all those lines of symmetry.
b) Sketch "$r = \sin(3\theta)$" and note the lines of symmetry on the graph.

B42. One of the most famous problems of geometry is to find a procedure that will trisect any angle (that is, divide it into three equal angles). It has been proven that it cannot be done with "Greek" rules, that is, using only a compass and straightedge. However, given a Spiral of Archimedes, any angle can be trisected. How? [By the way, the Spiral of Archimedes cannot be constructed with a compass and straightedge.]

B43. Suppose we wish to graph "$r = f(\cos(n\theta))$." For any f, there will be lines of symmetry of the graph because of the symmetry of cosine. Find all those lines of symmetry.

Section 8.4. Parametric Equations

Parametric equations are useful for describing motion. Motion produces change, and calculus is the mathematical subject which studies change. So, parametric equations are useful in calculus.

The usual type of graph is described in "functional" form, that is, y is given as a function of x by an equation of the form "$y = f(x)$." If this type of equation is used to describe a path of a moving object, two severe limitations leap to mind. One is that, because f is a function, the path can have only one y-value for each x-value, so the functional equation cannot describe a non-functional path such as in Figure 1.

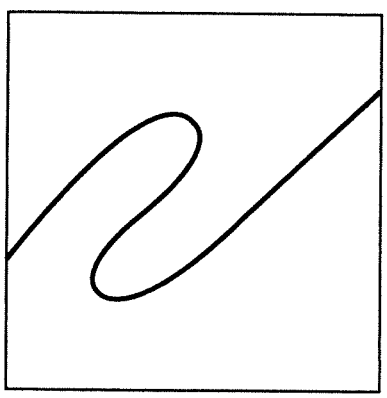


Figure 1: A path that cannot be described functionally by "$y = f(x)$."

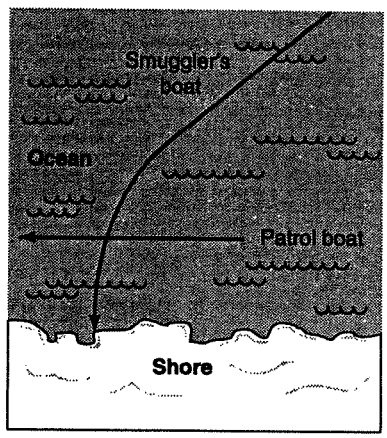


Figure 2: The routes of two boats. Do the boats meet?

A second limitation of a functional description of a path is that it does not contain information about *when* the object was at any point on the path. Figure 2 shows two paths, the route of a smuggler's boat from offshore to the coast and the route of a patrol boat. Will the patrol boat intercept the smuggler's boat?

We cannot tell. Figure 2 does not have information on *when* the boats will be at the various locations described by the paths.

Parametric equations use a parameter (often "t" for time) and two equations of the form "$x = f(t)$ and $y = g(t)$" to describe the (x, y) points on the path in terms of the parameter,

so that location and time are related. This contrasts with a functional description of the path in which the y-value is described in terms of the x-value alone.

Parametric Equations of Circles. There is more than one way to describe a circle. A circle centered at the origin with radius r has "standard form"

$$x^2 + y^2 = r^2,$$

according to 3.3.3. The so-called "functional form" requires y to be given in terms of x. Solving for y, the circle is given by

$$y = \pm\sqrt{r^2 - x^2}.$$

This is the form commonly entered into graphics calculators. Another approach is to describe x and y separately in terms of some parameter, say, "θ" or "t". Figure 3 reminds us that in polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$.

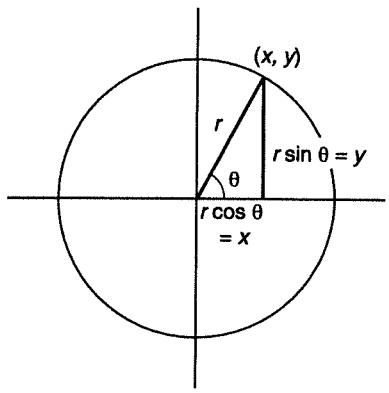


Figure 3: $x = r\cos\theta$ and $y = r\sin\theta$.

Example 1: Suppose a wheel of radius 5 rotates counterclockwise 1 radian per second. Describe the position at time t of a point on the circumference, if its initial position is at the rightmost point on the circle.

The speed of rotation and orientation are such that $\theta = t$ in the usual polar coordinate system. The radius is 5, so parametric equations for x and y are given by

$$x = 5\cos t \quad \text{and} \quad y = 5\sin t.$$

Figure 4 plots the path with the addition of labels relating position to time.

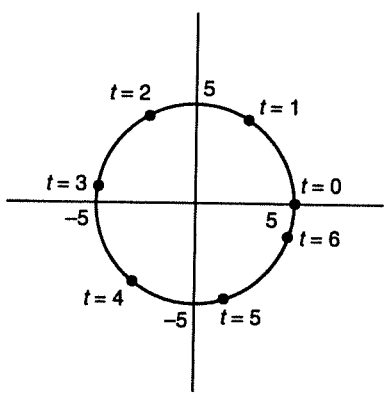


Figure 4: A circle of radius 5.
$x = 5\cos t$. $y = 5\sin t$.
[-10, 10] by [-10, 10].

Calculator Exercise 1: Learn how to use "Parametric" mode on your calculator. For example, plot the graph in Figure 4 and watch it develop as t increases (problem A1, B24).

Parametric equations give x and y in terms of t. It may be possible to convert parametric form to functional form by eliminating the t.

Example 1, continued: Let $x = 5\cos t$ and $y = 5\sin t$ be a parametric description of a path. If we wish to describe the path without reference to time, we can take the two equations and eliminate t.

Squaring x and y and adding the result,

$$x^2 + y^2 = (5\cos t)^2 + (5\sin t)^2 = 25(\cos^2 t + \sin^2 t) = 25.$$

Of course, "$x^2 + y^2 = 25$" is the equation of the circle with radius 5 centered at the origin.

(8.4.1) $$x = r\cos t \text{ and } y = r\sin t,\ 0 \le t < 2\pi,$$

are parametric equations of a circle centered at the origin with radius r.

Example 1, continued further: If we want a different speed than one revolution in 2π units of time, we can replace "t" with some function of t. For example, let

$$x = 5\cos(2\pi t) \text{ and}$$
$$y = 5\sin(2\pi t).$$

Now, as t changes from 0 to 1 the arguments of sine and cosine change from 0 to 2π, so one revolution occurs in 1 unit of time, which is quite a bit faster than in Example 1. The relationship between x and y is unchanged, but their relationship to time is changed (Figure 5). The increase in speed is easy to see as the picture develops on a graphics calculator (try it).

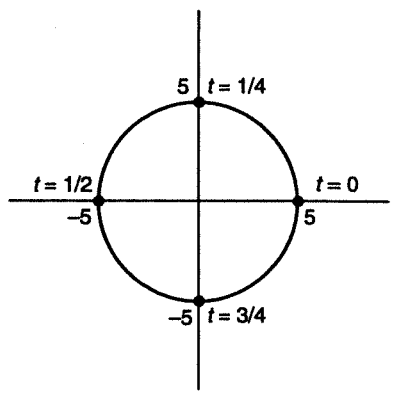


Figure 5: $x = 5\cos(2\pi t)$ and $y = 5\sin(2\pi t)$. [-10, 10] by [-10, 10].

More about Example 1: The previous parametric descriptions are of uniform motion about the circle. If we want to describe the same path, but with some sort of accelerating motion, we can replace "t" with some function of t, for example, "t^2".

Let

$$x = 5\cos(t^2) \text{ and}$$
$$y = 5\sin(t^2).$$

Again, the relationship between x and y is not changed. But the angle (angle t^2, playing the role of θ in polar coordinates) is increasing more and more rapidly (Figure 6).

time t	angle t^2	change in angle during the previous 1/2 second
0	0	
1/2	1/4	1/4
1	1	3/4
3/2	9/4	5/4
2	4	7/4

The rotation is speeding up.

In this example the "t" of Example 1 has been

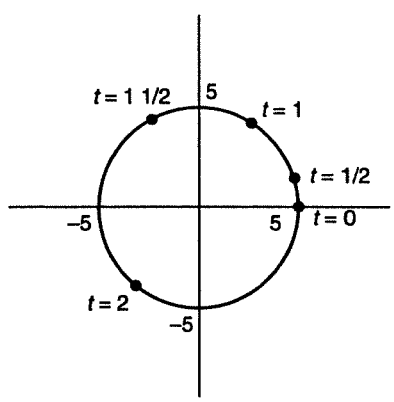


Figure 6: $x = 5\cos(t^2)$ and $y = 5\sin(t^2)$. [-10, 10] by [-10, 10].

replaced by "t^2" in both equations. If "t" is replaced *in both equations* by the same function of t, the graph will still be a circle. The relationships of x to time and y to time would change, but the relationship of x to y would not (problem B4).

For any function f,

(8.4.2) $$x = r\cos(f(t)), \text{ and}$$
$$y = r\sin(f(t)),$$

are parametric equations of a circle centered at the origin with radius r (problem B5) if the range of f includes the interval $[0, 2\pi)$. [Otherwise it might be just part of a circle.]

Example 2: Suppose a rod 15 centimeters long is pinned to a rotating wheel at a radius of 5 centimeters (point A, Figure 7). The piston (point P) at the other end of the rod slides vertically up and down. Describe the path of point P.

The position of P depends upon the angle θ in the figure. The motion is vertical, $x = 0$, regardless of θ. The vertical component of P can be determined in more than one way. We could use the Law of Cosines. Or we could treat the y-value as OC plus CP. By trigonometry, OC is $5\sin\theta$ and CA is $5\cos\theta$. Now, CP can be determined with the aid of the Pythagorean theorem: $CP^2 + CA^2 = 15^2$.

$$CP = \sqrt{15^2 - (5\cos\theta)^2}.$$

Therefore, adding OC and CP,

$$y = 5\sin\theta + \sqrt{15^2 - (5\cos\theta)^2}$$

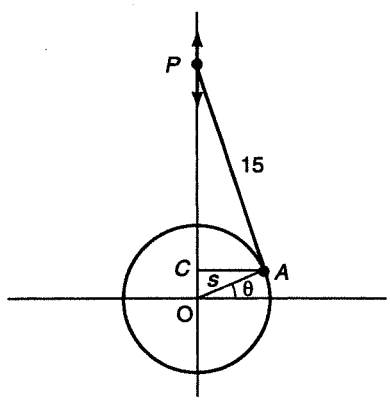


Figure 7. Point P, which moves vertically, is connected by a rod of length 15 to a point, A, that rotates with radius 5.

Now, suppose the wheel rotates 20 times a second. 20 revolutions is $20(2\pi)$ radians. So $\theta = 20(2\pi)t$. Parametric equations of the motion of P are given by

$$x = 0 \quad \text{and } y = 5\sin(40\pi t) + \sqrt{15^2 - (5\sin(40\pi t))^2}.$$

Figure 8 plots the vertical component of the motion as a function of time (problem A2).

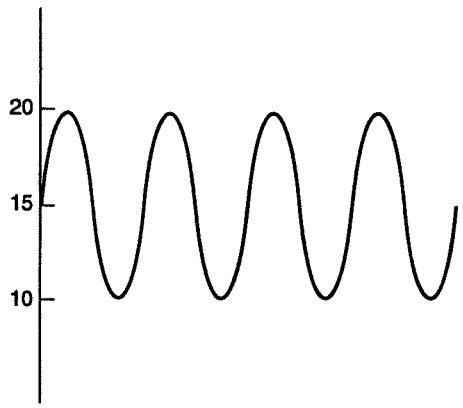


Figure 8: The vertical position of P in Example 2 as a function of time. [0, .2] by [5, 25].

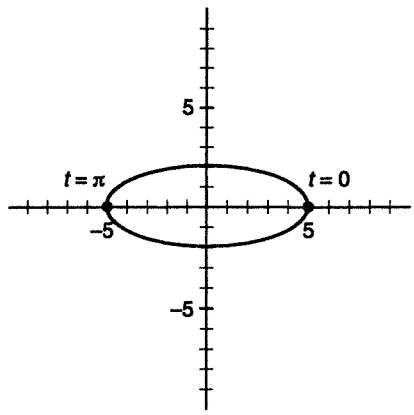


Figure 9: The ellipse described parametrically by $x = 5 \cos t$ and $y = 2 \sin t$. [-10, 10] by [-10, 10].

Example 3: Let $x = 5 \cos t$ and $y = 2 \sin t$. Graph the curve and find y in terms of x.

The graph is the ellipse in Figure 9. We can see that it is an ellipse by eliminating t. Divide by the constant and square.

$(x/5)^2 = \cos^2 t$. $(y/2)^2 = \sin^2 t$.

Adding, since $\sin^2 t + \cos^2 t = 1$, we obtain

$$\frac{x^2}{5^2} + \frac{y^2}{2^2} = 1.$$

which is the standard form of the equation of an ellipse (3.3.4) (problem B7).

Projectiles. The positions of projectiles are often represented parametrically. The vertical and horizontal components of the motion of ball or artillery shell can be treated independently. If air resistance is neglected, we can obtain simple answers. (Air resistance is an important factor, so neglecting it makes the answers wrong. Nevertheless, it is a good "first approximation" and the real effect of air resistance is complicated and cannot be derived mathematically until after calculus.)

Example 4: Assume that, due to the force of gravity, the vertical coordinate of a projectile is given by

$$y = y(t) = -16t^2 + 100t + 5.$$

This is the usual formula (from Example 3.2.8) with distance measured in feet, time measured in seconds, initial upward speed 100 feet per second, and initial vertical coordinate 5 feet above ground level at $y = 0$.

If the initial speed in the x-direction is 800 feet per second, without air resistance

the object would continue the same horizontal speed, so

$$x = x(t) = 800t,$$

from "distance equals rate times time," assuming the initial horizontal coordinate is zero.

We can use a calculator to plot the graph parametrically, but the standard window is not appropriate. Figure 10 shows the trajectory (path) in a suitable window.

When and where does the projectile hit the ground? Describe the path in functional form.

The projectile hits the ground when $y = 0$. So solve $-16t^2 + 100t + 5 = 0$ for t. Then plug that solution into "$x = 800t$" to find the corresponding x value (problem A3).

To describe the path in functional form, eliminate t. From the equation for x, $t = x/800$. Then t can be replaced by $x/800$ in the equation for y to give y in terms of x.

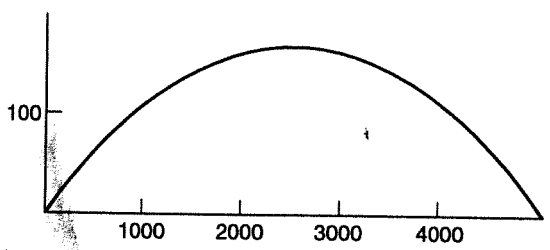


Figure 10: The path of a projectile. $x = 800t$. $y = -16t^2 + 100t + 5$. [0, 5000] by [0, 200].

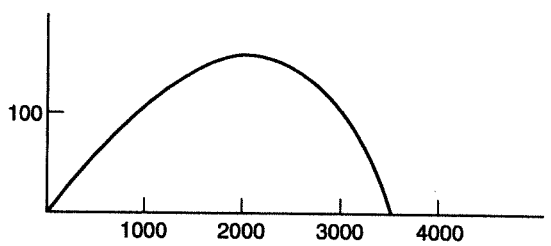


Figure 11: The path of a projectile. $x = 800t - 40t^2$. $y = -16t^2 + 100t + 5$. [0, 5000] by [0, 200].

Example 5: Suppose a projectile moves with the vertical motion described in Example 4,

$$y = -16t^2 + 100t + 5,$$

but the horizontal speed slows down as time passes so $x(t)$ is less than $800t$. For example, let $x = 800t - 40t^2$, $0 \le t \le 10$.

Plot the trajectory (problem A4).

Plot it parametrically (Figure 11). In this example it not so easy to eliminate t. But there is no real need to find or use a functional equation that gives y in terms of x. One of the advantages of parametric equations is that some curves are far easier to describe in parametric equations.

Parametric Equations of Lines. The point-slope formula for lines follows from similar triangles as discussed in Section 3.1 (Figure 3.1.3). Parametric equations for

lines follow similarly. Figure 12 illustrates two similar triangles.

Parametric equations for lines give x and y in terms of a particular point on the line (labeled "(x_1, y_1)" and "B") and two direction numbers (labeled "a" and "b") which are the change in x-value and change in y-value to any other particular point (labeled "A") on the line.

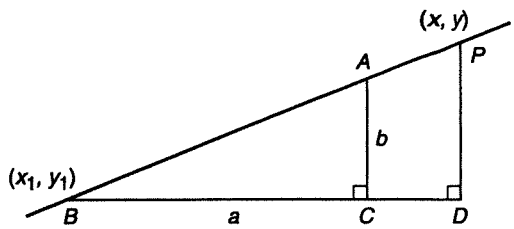


Figure 12: Similar triangles.

Let $P = (x, y)$ represent a general point on the line, and form the similar triangles in the picture. Now DB is some multiple of a. Call that multiple t, so $DB = at$, where "t" is the parameter (t looks to be about 1.3 in Figure 12). Then, by proportionality of sides of similar triangles, PD is bt. Therefore the coordinates of P satisfy $x = x_1+DB = x_1+at$ and $y = y_1+PD = y_1+bt$.

Therefore,

(8.4.3) $x = x_1 + at$ and $y = y_1 + bt$

are parametric equations of the line through the point (x_1, y_1), with direction numbers a and b and with parameter t.

When $t = 0$, the point is B. When $t = 1$, the point is A. If $a = 0$, x does not change and the line is vertical. If $b = 0$, y does not change and the line is horizontal. The slope of the line is b/a, if $a \neq 0$.

Example 6: Find parametric equations of a line through (1, 5) and (3, 2).

Sketch a figure (Figure 13). x changes 2 units when y changes -3 units, so let the direction numbers a and b be 2 and -3. Parametric equations (not the only ones) are:

$x = 1 + 2t$ and $y = 5 - 3t$.

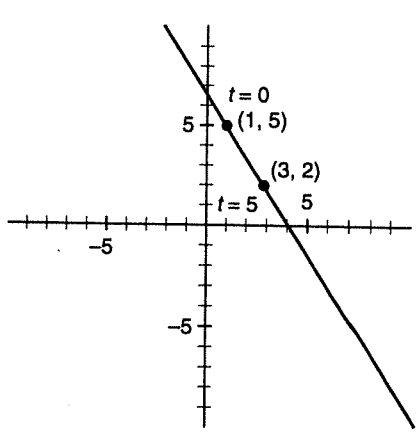


Figure 13:
$x = 1 + 2t$ and $y = 5 - 3t$.
[-10, 10] by [-10, 10].

Example 6, continued. Suppose an object undergoing uniform linear motion is at (1, 5) at time 0 and at (3, 2) at time 5. Find parametric equations of its path.

The points are the same as before, but we need to change the time scale. The equations from Example 6 yield (1, 5) when $t = 0$ as we want, but they yield (3, 2) at time $t = 1$, not at time $t = 5$. So, simply change the scale by a factor of 5.

$$x = 1 + 2(t/5) \text{ and } y = 5 - 3(t/5).$$

Regrouping, the equations are

$$x = 1 + (2/5)t \text{ and } y = 5 - (3/5)t.$$

By similar triangles as in Figure 12, we see that if a and b are direction numbers of a line, so are ka and kb for any $k \neq 0$. Here the direction numbers of the same line are 1/5 the direction numbers used in Example 6.

Formula 8.4.3 describes uniform motion along a line. If we want to describe non-uniform motion, simply replace "t" by some function of t (problem B3).

Inverses. Inverses of relations given parametrically are particularly easy to state -- simply interchange the expressions for x and y.

Example 7: The equations $x = t^3 - 5t$ and $y = 6 \cos(t - 4)$ yield the relation graphed in Figure 14 (solid line). Therefore the inverse relation is given by

$$x = 6 \cos(t - 4) \text{ and } y = t^3 - 5t$$

(Figure 14, dashed line). Of course, as we saw in Section 2.3 on inverses, the points of the inverse relation are the mirror image through the diagonal line $y = x$ of the points of the relation.

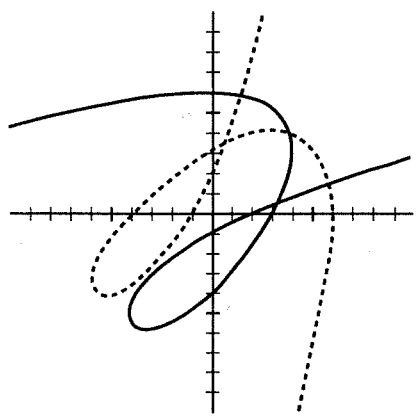


Figure 14: $x = t^3 - 5t$ and $y = 6 \cos(t - 4)$ [solid], and its inverse [dashed] [-10, 10] by [-10, 10].

Functional form can always be converted to parametric form.

(8.4.4) The points given in functional form by
$y = f(x)$ for x in some domain
are the same points given in parametric form by
$x = t$ and $y = f(t)$ for t in the same domain.

Using this idea, we can express and graph the inverse of any function given in functional form, even if its inverse cannot be expressed functionally.

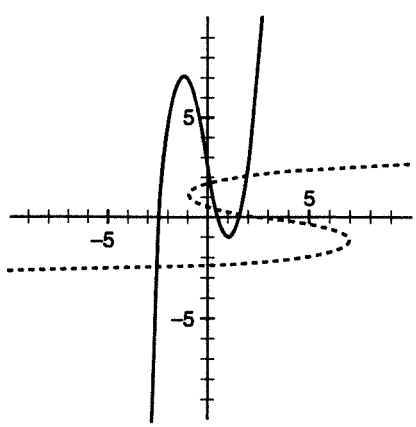


Figure 15: $y = x^3 - 5x + 3$ and its inverse [dashed]. [-10, 10] by [-10, 10].

Example 8: Let $y = x^3 - 5x + 3$ (Figure 15, solid line). Because some values of y (for example, $y = 2$) correspond to more than one value of x, the inverse is not a function and cannot be expressed

functionally. Nevertheless, we can plot the inverse on a calculator by thinking of the function parametrically as $x = t$ and $y = t^3 - 5t + 3$. Then its inverse can be plotted as $x = t^3 - 5t + 3$ and $y = t$ (Figure 15, dashed line).

Conclusion: Parametric equations have some advantages over equations in functional form. They can express non-functional relationships and they can relate positions to time. Also, some relationships that could be expressed functionally are easier to express parametrically.

Terms: parametric equations, direction numbers.

Exercises for Section 8.4, "Parametric Equations":

A1. Do Calculator Exercise 1. That is, use your calculator to plot the parametric equations in Example 1 (Figure 4), "Example 1, continued further" (Figure 5), and "More about Example 1" (Figure 6). The final graphs are the same. Comment on the differences in how the plots *develop* as your calculator plots them. [If your calculator plots the graphs too rapidly to see the difference in time of development, see problem B26 for a way to slow it down.]

A2. a) In Figure 7, what is the minimum possible y value of point P? b) At what value of θ does it occur? c) What is the maximum y value? d) At what value of θ does it occur?

A3. In Example 4, when and where does the projectile hit the ground? Describe the path in functional form.

A4. In Example 5, where does the projectile hit the ground?

^^^^ Identify the *type* (no details) of shape described by the parametric equations:

A5. $x = 2\cos t$ and $y = 2\sin t$.
A6. $x = 20\cos t$ and $y = 20\sin t$.
A7. $x = 6\cos(3t)$ and $y = 6\sin(3t)$.
A8. $x = 7\cos(e^t)$ and $y = 7\sin(e^t)$.
A9. $x = 5 - 2t$ and $y = 4 + 9t$.
A10. $x = 4 + t$ and $y = 12 + 5t$.
A11. $x = t^2$ and $y = 3t^2$.
A12. $x = e^t$ and $y = 5 - e^t$.
A13. $x = t$ and $y = t^2$.
A14. $x = 3t$ and $y = t^2 - 4$.
A15. $x = t^2$ and $y = 4t$.
A16. $x = t^2 + t$ and $y = t - 6$.
A17. $x = 2\cos t$ and $y = 3\sin t$.
A18. $x = 7\cos t$ and $y = 4\sin t$.
A19. $x = \cos t$ and $y = 3\cos t$. [Be careful.]
A20. $x = 2\sin t$ and $y = 5\sin t$. [Be careful.]
A21. $x = 5\sin t$ and $y = 5\cos t$.
A22. $x = 4\sin t$ and $y = 6\cos t$.

A23. Watch the two graphs develop on a graphics calculator to answer this question: What is the difference between the development of the graph of "$x = 5\sin t$ and $y = 5\cos t$" and the graph of "$x = 5\cos t$ and $y = 5\sin t$" (from Example 1)?

A24. Watch the two graphs develop on a graphics calculator to answer this question: What is the difference between the development of the graph of "$x = \sin t$ and $y = \sin t$" and the graph of "$x = \cos t$ and $y = \cos t$."

A25. Watch the two graphs develop on a graphics calculator to answer this question: What is the difference between the development of the graph "$x = 1 + 2t$ and $y = 5 - 3t$" for all t (from Example 6) and the graph of "$x = 3 - 2t$ and $y = 2 + 3t$" for all t [where the other point is treated as (x_1, y_1) and the negatives of the direction numbers are used]?

A26. Watch the two graphs develop on a graphics calculator to answer this question: What is the difference between the graph of "$x = 5 \cos t$ and $y = 2 \sin t$" (from Example 3) and the graph of "$x = 5 \cos(t - \pi)$ and $y = 2 \sin(t - \pi)$".

^^^^ Find parametric equations of
A27. A line through (-1, 6) and (2, 4).
A28. A line through (5, 10) and (1, 2).
A29 A circle with radius 6 centered at the origin.
A30. A circle with radius 2 centered at the origin.
A31. An ellipse centered at the origin with $a = 2$ and $b = 8$.
A32. An ellipse centered at the origin with $a = 10$ and $b = 3$.

^^^^ Give the slope of the line.
A33. $x = 5 + 3t$ and $y = -7 - 2t$.
A34. $x = -2 + 6t$ and $y = 4t$.
A35. $x = 12 - 3t^2$ and $y = 20 + t^2$.
A36. $x = -1 + 2e^t$ and $y = 9 + e^t$.

^^^^ Convert the parametric equations to functional form.
A37. $x = 4 + 3t$ and $y = 2 - t$.
A38. $x = 5 - 6t$ and $y = 2 + 3t$.
A39. $x = 7 \sin t$ and $y = 7 \cos t$.
A40. $x = 5 \cos t$ and $y = 3 \sin t$.

^^^^^^^^

B1.* Give two advantages of parametric equations over equations for y in terms of x ("functional" form).

B2.* a) Relate the "m" of "$y = mx + b$" to the "a" and "b" which are direction numbers in 8.4.3 of the parametric equations of that line. b) Does a given line have a unique a and b? Explain.

B3. Formula 8.4.2 is more general than Formula 8.4.1, because the motion in 8.4.2 need not be uniform. Generalize Formula 8.4.3 the same way, that is, generalize the formula for parametric equations of a line to the case of non-uniform motion.

B4. Prove that equations 8.4.2 yield a circle.

B5. Equations 8.4.2 are introduced with the phrase "For any function f such that $[0, 2\pi)$ is a subset of its range." a) Why is the range of f relevant? b) Give a (very) simple example of an f and its domain such that the equations 8.4.2 do not yield a complete circle.

B6. Give parametric equations of a circle centered at (h, k) with radius r.

B7. Inspect Example 3 and then give parametric equations for *general* ellipses centered at the origin with horizontal and vertical semi-axes of lengths a and b (corresponding to Formula 3.3.4). [Continued in B8.]

B8. [After B7]. Give parametric equations for general ellipses centered at (h, k) with horizontal and vertical semi-axes of lengths a and b (corresponding to Formula 3.2.11).

^^^^ Find parametric equations of a circle centered at the origin with radius 5 such that
B9. $t = 0$ corresponds to the leftmost point on the circle.
B10. $t = 0$ corresponds to the topmost point on the circle and 1 revolution occurs when $t = 1$.

^^^^ Find parametric equations of uniform motion along a line such that
B11. $t = 0$ corresponds to the origin and $t = 10$ corresponds to (1, 5).
B12. $t = 0$ corresponds to (0, 3) and $t = 2$ corresponds to (2, 1).

^^^^ Convert the parametric equations to functional form and identify the type of shape of the graph.
B13. $x = 3 \sin t$ and $y = 7 \sin t$ [consider the domain.]
B14. $x = 5 \sin(t^2)$ and $y = 10 \sin(t^2)$ [consider the domain.]
B15. $x = 50t$ and $y = -9.8t^2 + 40t$.
B16. $x = t - 5$ and $y = t^2 + 4t$.
B17. $x = e^t$ and $y = 2e^t$ [consider the domain.]
B18. $x = t^2$ and $y = 5t^2$ [consider the domain.]

B19. Suppose a projectile moves with vertical component given by $y = -16t^2 + 200t + 50$ and horizontal coordinate given by $x = 2500t - 300t^{1.5}$. If ground level is $y = 0$, where does it hit the ground?

^^^^ Find the equations for graphing on a calculator the inverse relation of
B20. $f(x) = x^3 - 4x - 2$.
B21. $f(x) = \cos x$, all x.
B22. $f(x) = 5 \sin x$, all x.
B23. $f(x) = x^4 - 5x^2$.

B24. This section discussed parametric equations of lines in the two-dimensional plane. The idea of lines in three-dimensional space is precisely similar. We need three direction numbers (instead of two) and a point that the line goes through. The equations are

$$x = x_1 + at, y = y_1 + bt, \text{ and } z = z_1 + ct.$$

Find parametric equations of a line through (1, 2, 3) and (4, 6, 8).

B25. Give parametric equations for any graph that somewhat resembles Figure 1.

B26. Learn how to adjust the speed at which graphs develop on your graphics calculator. On some models, in "parametric" mode, the window includes the domain of t and a "Tstep" entry which can be used to adjust how much t is advanced between evaluations. If the advance is smaller, more t-values are used and the graph develops more slowly. Here is the problem: Graph $x = 8\cos(t^2)$ and $y = 8\sin(t^2)$ for $0 \le t \le 2\pi$. a) What is the shape graphed? b) How many times around is the shape traced? c) On many calculators, the resulting shape is a bit (or a lot) fuzzy. Why? d) What does this have to do with the amount t is advanced between calculations?

B27. Find parametric equations of the path of a moon revolving around a planet which is revolving about a sun. To make the graph fit on your calculator screen, assume the distance from the planet to the sun is 8 times the distance from the planet to the moon, and assume the moon goes around the planet 12 times during one revolution of the planet around the sun.

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