Instructor's Manual

For

Precalculus

Sixth Edition

by Warren W. Esty

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for *Precalculus, Sixth Edition,* by Warren Esty

Why read this manual? This manual is intended to rapidly teach you, the instructor, some of what you would know about the course if you had already taught it. It reflects the experiences of dozens of previous instructors and will help you prepare a class that addresses the needs of your students.

This manual
1) provides section-by-section comments,
2) notes what to emphasize and what to assume students already know,
3) lists problem numbers of good problems to ask students to answer aloud in class (they are marked with a smiley face, ☺, in the text).

Goals. Make sure you and your students read the preface (page v) which lists ten goals of the course. The goals are more ambitious than the goals of many similar courses, particularly in the area of developing the student's ability to read mathematics with comprehension.

Audience. This text is designed for an audience of students who intend to take calculus or major in some area in which mathematics is important and have taken Algebra II

Lectures and homework. At a university, students are supposed to work two hours outside class for each hour in class. Remind students of this. Do not expect to be able to cover everything in the text in a 50-minute class. Skip the easy material they are supposed to know from prerequisite courses (Algebra I, etc.) and cover the new material, or cover the "old" material with the new, higher, level of sophistication. The section-by-section comments will help you with this. Almost every section has concepts and skills treated at both a basic prerequisite level and at a higher level. After rapidly reviewing the basics, it is appropriate to move up to the higher level of the "B" homework problems.

Encourage students to read
1) in order to learn to read (they learn to read math by reading math), and
2) in order to learn (wouldn't it be great if they could read well enough to learn math outside of class?).

I recommend you feel free to lecture on some examples straight from the text. Of course, you may invent your own examples, but there are enough in the text that there will still be a lot for students to read even if you use some examples straight from the text. Besides, many students do not have an easy time reading any math text and you can help smooth the way by doing some of the examples in class.

If you happen to finish your lecture early, you can skip straight to the homework section and begin to do problems. There are more than enough problems, so feel free to do some in class.

Some problems are actually designed to be done aloud in class. They are marked with a smiley face: ☺. Such problems are noted in the comments on the sections and I recommend that you ask some of those questions of particular students in class (I don't recommend you ask the class as a whole – the students who need to be forced to think will not participate if not individually asked to.) For example, at the end of the lecture on Section 1.1, students should be able to do B17-40, and B5-B15 aloud. Asking them to respond in class shows them that knowing how to do such problems is important, and that we expect rapid fluency in such problems, not an uninvolved "I can look it up" attitude.

Significant digits: Many problems have their answers given with two significant digits in the text right alongside the problem. That way, students can get immediate feedback about whether they did the problem right. However, students are required to supply their answers with three or more significant digits so we can see that they actually did the work. Some students who don't actually do the work will hand in two-significant-digit answers copied from the text or (student) Solutions Manual. I recommend that you set the proper tone **early** in the course by immediately rejecting papers with answers with only two-significant digits (significant digits are discussed in Section 1.2). Otherwise, students will test your resolve again and again by taking the time-saving shortcut of copying two-digit answers on some problems and you will be stuck with repeatedly deciding how to treat their (lack of) work.

Chapter 1. Reading Mathematics

Chapter 1 is unlike any other text's introductory chapter. Chapter 1 is designed to orient students so that they can learn and retain math efficiently. Read it all first so you can see where it is leading.

Chapter 1 emphasizes concepts students must have to understand and retain algebra, and **to be able to learn math by reading it**.

The immediate purpose of Chapter 1 is to

1) **Change the students mind-set away from thinking that algebra is about numbers to understanding that it is really about operations and order**. The focus is strongly on operations and order.
2) Show that algebraic notation is designed to express thoughts about operations and order, and show how it does it (Sections 1.1-4).
3) Remind students of the importance of order of operations (All sections, but especially 1.2).
4) Introduce terminology and notation that describes essential concepts of algebraic thought (Sections 1.3-4).
5) Show how mathematical theorems and identities express abstract patterns of operations with placeholder and explain how they can tell you what to do when your problem matches a given pattern (Section 1.4).
6) Above all, teach students how to *read* mathematics with comprehension (Section 1.4).
7) Familiarize them with the power of a graphics calculator (for solving as well as for graphing) (Section 1.5).
8) Organize ways to solve equations into four manageable categories (Section 1.6).

This introductory chapter is titled "Reading Mathematics." To read symbolic with comprehension, what concepts are essential?

1) *operations* and *order* (not numbers) are the conceptual objects of algebra.
3) *placeholders* are used to express thoughts about operations and order (They are fundamentally different than "unknowns").
3) *expressions* and *equations* are related but different
4) *identities* and other *theorems* express *problem-patterns* and *solution-patterns* in symbolic notation.

Chapter 1 is written with the typical precalculus student in mind. Almost all of the students intend to take calculus and have already taken three or four years of algebra and trigonometry. Yet many still do not command the facts and methods of those subjects. Why not? And, what can be done about it?

Many precalculus students think that, with a bit of review, they could already do the math in this text. Some are right, and they belong in calculus. But many are mistaken. Experience shows that most students who place into precalculus are familiar with the material only in a vague and unsatisfactory manner, and that, in fact, they cannot do a wide variety of types of problems that are essential prerequisites for calculus unless those particular types of problems have been reviewed very recently. These students apparently have developed an effective *short-term* memory of mathematics (so they can pass unit exams in school), but have little long-term retention. They will not succeed in calculus until their mind-set about how mathematics is learned and retained has changed dramatically. Part of the purpose of Chapter 1 is to reorient these students.

Calculators: Computation with calculators is important beginning in Section 1.2 and graphing with calculators is important beginning in Section 1.5.

Section 1.1. Algebra.

This section illustrates the purpose of algebraic notation. Many students do not use algebraic notation effectively.

When the words in a word problem, or symbols in an expression, or a basic formula suggest operations you actually *do* (as in Example 8A), the problem is "direct" and not really algebraic. When the words or a formula suggest operations you do not do, but *represent* in symbolic notation (as in Example 8B), the problem is "indirect." The operations you end up doing are not the ones given in the problem, which is why 8B requires algebra. Direct problems are closer to arithmetic than algebra. Only indirect problems are truly algebraic. Mention and distinguish these two terms, "direct" and "indirect."

Most students are far better at direct than indirect problems. You can compare Examples 8A and B to make the point about the use of symbolism to express a sequence of operations.

Personally, to make this point, before I lecture, I give an ungraded word-problem quiz with problems parallel to Examples 8A and B to illustrate what I mean. Then my lecture can be about the quiz and how it relates to operations and order (as opposed to just numbers) by comparing the two types of problems. The quiz emphasizes the point I could make with Example 8. If you give such a quiz, you will find that many students cannot even set up the equation. This leads to the point of the section. Then do the rest of the examples. This is not a long section.

[No emphasis on graphing until Section 1.5.]

Examples: Any examples similar to Examples 1 and 2. Example 8.

Aloud in class: Any from A1-6, A15-18, B5-40.

Section 1.2. Order Matters!

We require answers to be given with at least three significant digits. Explain what "significant" digits are and then go over some problems like A2-A6 in class by asking particular students to give the answer aloud. This is *not* the main lesson of the section.

Algebra is about operations and order and this section shows that order matters. Most students already know the conventions of algebraic order (at least, in theory). Emphasize

1) The three grouping symbols available in written math that cause trouble with calculators
   a) extended fraction bars
   b) extended square root bars
   c) exponents as superscripts [less important]
2) the difference between -5^2 and $(-5)^2$.
3) how to enter expressions as complex as the Quadratic Formula
   [The keystroke sequence for the Quadratic Formula is non-trivial. They may program their calculator to solve quadratic equations, but they still need to know how to use their calculator correctly (Consider problems like A13-32 and B2-16. If b is negative, many students have trouble entering the "$b^2 - 4ac$" part correctly.)]
4) You might emphasize pronunciation. It may be a factor in why students enter expressions incorrectly. $(a+b)/c$ may sound like $a+b/c$. How can we expect students to work comfortably with sentences like A36-A65 if they can't even say them aloud (or, in their mind)?

Examples: Strong emphasis on calculator computation. Possible classroom examples: All Calculator Exercises. -3^2 vs. $(-3)^2$. New quadratic formula examples. We are happy if they have, or program in, a Quadratic Formula program in their calculators.] Homework B2-B16 are complicated and good if time is left over.

Aloud in class: A2-6, 7-9, 36-65. Any from A12-32 and B2-B16, or problems similar to them, can be asked of the whole class. They will take some time, but not too long. **Involve the students**. Their learning depends upon *their* work, not so much on yours!

Section 1.3. Functions and Notation. Emphasize

1) the way to solve an equation is dependent upon the sequence of operations, not the numbers (Example 2).
2) identities have information about operations and order, not about numbers (Example 1)
3) "function loops" can illustrate the importance of order and inverses (Example 2).
4) functional notation with "$f(x)$" uses of placeholders. Emphasize the term "placeholder" (a synonym is "dummy variable.")
5) the sequence argument-function-image in that order. Use the terms "argument" (input) and "image" (output) when appropriate.
6) thinking of "f " as a command, distinct from "$f(x)$" which is the image, representing a number, not a sequence of operations. This same function applies to expressions other than "x". (Examples 4, 8-10.)
   Emphasize conceptualizing the *rule* itself. Distinguish it from the *notation* for the rule. Note the use of *placeholders* (p. 23, 25) to "carry" the operations.
7) natural domain, domain (Example 15 is good because its domain and range are non-trivial. It also lets you use the graphing calculator early on in the course if you want to use it right away.
   [square roots, division by zero, and certain other functions restrict the natural domain (e.g. log)]

We want students to be able to read math! A grasp of placeholders is essential to reading math. Functions are great topic for discussing how to read, because functions use placeholders in a critical manner.

Examples: Graphing is useful, especially in Example 15, but graphing is not emphasized until Section 1.5. You can skip graphing here, although I would do it. Possible classroom examples: Ex 1, 2, 8, 9, 11, 13, and 15 (an excellent example, discussed next).

State Example 15, or some similar problem, and then pause and let students work on it. Some students will not think to build a formula using algebraic notation. Some will need help with the basics of using their graphics calculators. (They will get lots of help in Section 1.5.) Some will have trouble finding a good window for the graph. This is a good opportunity to discuss domains and ranges, because the "standard" window will not do.

They should get the area formula (in terms of the distance the pen projects from the old fence) $A(x) = x(100 - 2x)$. (Note how this expresses the *evaluation* process.) To maximize this, graph it. However, the domain and range are relevant. The "standard" window will not do. Ask what values of x are possible. $0 \leq x \leq 50$ (or else there is no pen). What values of y should appear on the graph? Values up to 1250 (square feet) are possible, so I use $0 \leq y \leq 1500$. Now it is possible to see a "representative" graph and do the maximization.

If the students can identify the domain, the calculator will help them with the y-interval. I use *trace* to find x-y pairs and then adjust the y-interval accordingly.

Aloud in class: A3-20. (After they understand it: B9-12, B3-8, B16-20.)

Section 1.4. Reading and Writing Mathematics.

We want students to be able to read math! This section is about **how to read mathematics to learn mathematics**. The goal of this section is to improve reading and writing skills by discussing the grammar of the language and holding them responsible for some basic language concepts.

This section opens the student's eyes to the connection between methods and written facts in mathematics. The rest of the text, of course, uses modern methods of expressing mathematics which state methods as facts. But students are more comfortable with methods stated as commands: "Do this!" To become good at math, students need to know how to read facts. They like this section, but, of course, they cannot fully grasp it the first day. Some students may even need weeks to fully understand the connection between *doing* (methods) and *facts*. Emphasize

1) the three major ways to express methods:
   a) Formulas, [e.g. $C = \pi d$ or $A = \pi r^2$]
   b) Identities (equivalence of expressions),
   [e.g. $1a = a$. $a + 0 = a$. $(a/b)(c/d) = (ac)/(bd)$, $(a/b)/(c/d) = (ad)/(bc)$.]
   c) Relations between equations. (A sentence that asserts how two related equations are related.) [e.g. $x + a = b$ iff $x = b - a$. The Quadratic "Formula" is only the solution-pattern of the entire Quadratic Theorem which includes the "$ax^2 + bx + c = 0$" problem-pattern.]

Do this by abstracting the patterns after displaying several examples.
In each case, note the "problem-pattern" on the left and the "solution-pattern" on the right.

Important: They know, or should know, the methods. We are **not** teaching the methods here. We are teaching **how they are written**. Make sure your students know we are making this distinction.

2) the terms *placeholder*, *problem-pattern*, and *solution-pattern*.
3) the distinction between "holding a place" and "representing" a number.
4) the distinction between equality and equivalence
5) the relation between processes expressed as commands and as statements of fact. [We are teaching them to read and write.]

If someone asks if they have to know how to solve log or exponential problems (thinking that memorization of Theorem 1.4.8 may be required), say

> "We want you to learn how to read. This is just a sample theorem. If you are given that theorem, you should be able to read it and use it appropriately. But we won't require you to know log and exponential facts by heart until Chapter 5. However, on quizzes and exams we might give you this or some new fact that expresses a method, and you will be expected to be able to read the fact and use it. If we want you to algebraically solve problems with logs and exponents, we will give you this fact on the exam or quiz."

Examples: No emphasis on calculators. Example 10 (or any HW B37-60), 11 (or any HW B29-36), 14 (or any HW A3-6, B5-16), 15 (for HW A15-22),16 (for HW 53-54).

Aloud in class: A3-6, A7-10, B5-20, B37-60 (with your guidance), B71-76 (not so rapidly). B29-36, with time **and your help**, leading them through a numerical example, then to problem-pattern and then to the solution-pattern and connective.

Section 1.5. Graphs.

Most students know the basics of graphing, but they need to learn how to use the power of their calculator. Key ideas are

1) window
2) *zoom*
3) *trace*
4) representative
5) pixel, artifact (try $\sqrt{(83 - x^2)}$, $1/(1 - x)$)
6) expression
7) equation, solve

The *trace* feature treats a function as a set of ordered pairs than you can display. To go from x to y is to "evaluate an expression", to go from y to x is to "solve an equation".

8) graphs without expressions

[Very many students cannot read a graph to "Solve $f(x) = 1$" and find all solutions if there is more than one solution. Even more cannot read a graph to "Solve $f(x) > 0$." Students must be taught how to "Solve $f(x) = x$" given the graph of f. They can sketch in the graph of $y = x$ and find the x-value(s) where the two graphs intersect. Many students seem to think the solution to "$f(x) = 1$" is a geometric point and we need to emphasize it is the x-value of the point, not the point itself.]

Examples: Very strong emphasis on graphing. Calc Exercises 2-8, graph $1/(x - 1.5)$ for an "artifact", graph $\sqrt{(83 - x^2)}$ for another (the top half of the circle is incomplete), Example 6. Homework B17-25 can be done by students in class.

This is a lecture in which you must involve the students. Make them get out their calculators and actually do some of the examples themselves. Wander around among them, learn names, and look at how they are doing. Don't let some not participate.

I put an equation such as "$y = (x - 20)(x - 50)$" and its graph in a good window on the chalk board and ask them to play with their calculators until they can identify the window I used.

Aloud in class: A7-9, A10-11, B3-B8. Consider taking time for students to use their calculators to do some of B17-25. (There will be more – and more difficult – examples like these in Section 2.1.)

Section 1.6. Four Ways to Solve Equations.

When given an equation to solve, how do you decide which method to use? Well, there are only four basic methods. Learning their "format requirements" should give students some guidance.

This is a "compare and contrast" section, not a "how to" section. Your students should already know *how* to solve many equations. Emphasize *when* to use the various methods by comparing and contrasting them. Put up two similar equations that require different methods and teach how to *distinguish* the methods, but do not spend much time actually solving simple equations your students already know how to solve.

Emphasize

1) how to recognize what to do to solve an equation. Comparing and contrasting the methods – problems that look similar superficially but require different techniques (Homework A15-26, 27-46, B9-23, B3-8, B25-42 B43-68. These can be done aloud in class after lecture)
2) format requirements
   a) inverse-reverse

b) Zero Product Rule statement and format requirement
(Why do we "factor" expressions? To satisfy the ZPR format requirement.)
c) mention the Quadratic Formula (including the format requirement of the form "$ax^2 + bx + c = 0$"), but do not spend lots of time on it.

3) guess-and-check (as a non "algebraic" method. We use the term "algebraic" to distinguish traditional, algebraic, indirect, methods from guess-and-check which is a direct method).
4) changing the order of operations if the pattern is not yet right [Examples 2, 8, 11-13.]
5) discuss "direct" and "indirect" as in (3) above to help illuminate what algebra is.

Algebra is distinguished from arithmetic by its emphasis on indirect problems.

[Note: Algebraic notation expresses operations in a certain order. If, to solve a problem, you do those operations in that order, the problem is "direct." For example, "Find $f(5)$ when $f(x) = x^2 + x$." But most algebra is indirect. For example, to solve "$x^2 = 15$" you do not square 15. All the "algebraic" methods are indirect, but guess-and-check is direct. "Evaluation" is usually direct; "solving equations" is usually indirect.]

Examples: Minor emphasis on graphing. Possible classroom examples: Examples 1, 2, 4, 6, Thm 1.6.2 (this is for reading practice, not yet to memorize), 7 (very good), 9-14.

Aloud in class: A15-26, 27-46, B9-23, B3-8, B25-42 B43-68.

Chapter 2. Functions and Graphs

Section 2.1 has more about graphing – especially the visual effects of changing windows. Section 2.2 is on composition of functions and graphical interpretations of shifts and scale changes. Section 2.3 is on inverses of functions. Graphing calculators play a big role in Chapter 2.

Section 2.1. Functions and Graphs.

This section has a strong emphasis on graphing. Emphasize

1) Functions as sets of ordered pairs (as displayed by the *trace* feature).
2) How to read graphs. A graph displays ordered pairs (x, y) which contain information for **evaluating expressions** (finding $f(x)$ given x) and for **solving equations** (given y, finding x's such that $f(x) = y$).
3) How to select or adjust the window.

Do not just lecture. Be sure to involve the students by asking them to actually grab their calculators and do some of these problems.

Use $y = x^2$ to show that different windows make the graph appear different, but do **not** use it to make the wider/narrower or taller/shorter distinctions. The reason is that the example is too specific -- in it narrower is also taller! Shorter is also wider! The distinction between horizontal and vertical is not clear using x^2.

Use 5- x^2 instead, or my type of example in Figures 3A-C. Using 5 - x^2, it crosses the y-axis at 5 so you can see it look taller or shorter by changing the y-interval. Also, it crosses the x-axis so you can see its apparent width.

Examples: Figures 1A-C and Calculator Exercise 1 (different windows -- be careful, do **not** use this to illustrate taller-shorter-wider-narrower), CE2, CE3, (adjusting the window), 4. Do some of A1-6, B7-9 in class. Some problems like B10-24 on adjusting the window are good in class, letting students try first.

During class: After some lecture, tell them to "Do (any one of B10-16, or one of your choice like these)" and wait for them to try. Circulate and help. Or, ask them to make their picture of, say
$y = x(x - 5)(40 - x)$ look like the picture in B11. [-10, 50] by [-10000, 10000] comes close.

Aloud in class: After a lecture they should be able to answer B1-2 and B3-6 in class.

Section 2.2. Composition and Decomposition.

There is a strong emphasis on graphing. Also, we emphasize proper use of notation. Emphasize

1) two-stage functions may be built by successively applying two simpler component functions
2) decomposing two-stage (or three-stage) functions
3) getting the notation straight (which requires mentally separating the function from the notation). When $f(g(x))$ is given, what is f? What is g?
4) graphical effects (shifts and scale changes)
   [Changes in the argument cause the most difficulty. Why would $f(x - 4)$ have a graph 4 units to the *right* of the graph of $f(x)$? Why would sin(2x) be *half* as wide as the graph of sin x?]

Note: Horizontal scale changes are very useful. Be sure to emphasize how and why changes to the argument affect the graph.

Project a basic graph, say, x^2 or $|x|$ or sin x, on the overhead. Then go to the "y =" menu and let the students see you add in a similar expression with the argument or image changed. (x^2 and $|x|$ are not good for comparing $f(x)$ and $f(2x)$ and $2f(x)$, but they are excellent for comparing $f(x)$ and $f(x+3)$ and $f(x) + 3$. Then, before projecting the modified version, ask students to guess what the new graph will look like. ["up 6" or "3 to the left" or "twice as tall" or "half as wide," etc. By the way, quadratics, lines, and absolute values are **not** good for illustrating vertical scale changes, because since a "taller" parabola may be interpreted as "narrower"]. Use something like "sin x" or "x^3 - $5x$" for vertical changes ($f(x)$ and $f(2x)$ and $2f(x)$).

Examples: All, but especially composition where the argument changes, for example, Ex 14-17.]

Aloud in class: After lecture, A9-10, 11-12. Second day: B1-10, B43-46, B63-74 (harder), B83-88 (harder) .

Section 2.3. Relations and Inverses.

This section is mostly about tricky inverses, that is, inverses when the function is not one-to-one. Many students solve "x^2 = 16 with "x = 4," forgetting the negative solution. Later, in trig, many will solve "sin x = .5" and obtain only the acute-angle solution, when there is an obtuse-angle solution as well. Our objective is to help them recognize and remember when solving "$f(x) = c$" is not trivial.

In this section there is quite an emphasis on reading theorems.

For the inverse-reverse method of solving equations, we need to know how to "undo" operations. Some are easy to undo (for example, "Multiply by 3" is undone by "Divide by 3") but others are more complex. This section is primarily about the more complex ones. Emphasize

1) If we regard f as an evaluation *process* (rule) for finding the expression $f(x)$, then the solution *process* for solving the equation $f(x) = y$ is called "f inverse" and denoted f^{-1}. It is an abstract step up from solving "$f(x) = 5$" (and getting a number) to solving "$f(x) = y$" (and getting an expression which represents a process as well as a number). To find f^{-1}, solve $f(x) = y$ for x to find the solution process expressed in terms of y: $f^{-1}(y)$ (which represents a process, as well as a number). This is nicely illustrated with a "function loop" diagram. To express f^{-1}, switch letters if you want (to get $f^{-1}(x)$ instead of $f^{-1}(y)$ – they are just placeholders anyway). The *process* should parallel the process for solving "$f(x)$ = 5," but students find it much harder to deal with "y" than "5," particularly in equations such as $x/(x - 4) = y$ where they find it hard to isolate "x".
2) inverses that are not functions (for example, inverse relations for $f(x) = x^2$ and sin x). How are inverses defined when there is more than one x for a given y? For a given y, your calculator provides one x value as "the" inverse. How are the rest of the x's generated from the one your calculator provides? Sin x is a good example of this because sin $x = c > 0$ has <u>two</u> important solutions in the Law of Sines of trigonometry (Chapter 6.3). In degrees, your calculator provides one between 0° and 90°, but there is another important solution between 90° and 180°.
3) The graph of f^{-1} just reverses the roles of x and y in the graph of f.
(for example, e^x and ln x can illustrate this)

Examples: Some emphasis on graphing. Examples 2, 4, 6 (sine is particularly interesting because of its complexity, most other common functions are too simple to illustrate the idea of using one solution, given by the inverse function on your calculator, to generate others. I make them read and use Theorem 2.3.6). Theorem 2.3.7. 2.3.8. Graph a few like Figure 10 [use a **square** scale “Zoom 5", not “Zoom 6.”]

Aloud in class: After a lecture, A5-12. (B5-8 are harder to do in one’s head.)

Chapter 3. Fundamental Functions

Students have seen the fundamentals in previous courses. Our goal in Sections 3.1-3.4 is to **help them recall and retain the fundamentals**, especially as they apply in calculus contexts. Our goal in Sections 3.5-6 (word problems) is to help them **learn how to use algebraic notation properly in word problems**.

The text repeats the basic material about lines and quadratics, as is obviously necessary. However, in class the more-sophisticated material should be emphasized. We must presume they can read and review the most basic facts about lines, quadratics, and factoring by themselves. To promote retention we emphasize memorable pictures, explanations, and illuminating higher-level examples.

Use class time to emphasize activities designed to promote retention or designed to extend the concepts to advanced applications such as will be seen in calculus. Review the basics as they come up naturally in the advanced examples.

Section 3.1. Lines.

Most students have studied lines a lot already. Although the text develops the entire subject from the beginning, the classroom emphasis ought to be on retention and more sophisticated uses of lines. Emphasize

1) Figure 5 (as a memory aid to the two-point and point-slope formulas, which are otherwise easily forgettable messes of symbols)
2) the point-slope formula (Ex 3, 4) in preference to the slope-intercept formula (Ex 11)
3) approximation of curves by lines "linear interpolation" (Ex 14, 15, 16, 17)

A key idea of calculus is that curves may be approximated by lines, at least locally. The “local” importance of lines is why we prefer “point-slope” to “slope-intercept” form.

4) lines through points expressed functionally or with letters (e.g. Ex 13)

For reading and writing math:

5) what is a "parameter"?
6) what does "proportional" mean? [Many students have the vague idea it means “increase together,” but the term is more precise than that. For example, the area and side of a square “increase together,” but they are not proportional. Their relationship is not “multiply by a constant.”]

Examples: Two-point formula examples: 11, 13, 14 (long) and higher. "parameters".

Aloud in Class: A3-4, A5-10, A11-14, A41-44.

Section 3.2. Quadratics.

Most students have used the Quadratic Formula a great deal and can easily solve quadratic equations if the coefficients are numbers (as opposed to unusual letters or more complex expressions). The emphasis ought to be on more-advanced understanding of quadratics.

1) symmetry of the graph (Ex 1, 2, Figure 6)
2) location changes (Ex 2 and following)
3) completing the square (which relates to conic sections, including circles). This also relates to the derivation of the Quadratic Formula using inverse-reverse. The idea of completing the square is to rewrite the expression with only one appearance of "x" (as in the inverse-reverse format requirement) (Ex 2, 3, Fig. 6)
4) how the terms of the Quadratic Formula are evident in a graph (Figure 6)

5) when the quadratic applies and what *a*, *b*, *c* really are: Ex 7 and Calc Exercise 1, which is hard for them, but very illuminating. Homework B21-22 are similar to Calc Exercise 1.
6) maximum-minimum problems (e.g. the complex Ex 9)

For reading and writing math:

7) state the (entire) result commonly called the "Quadratic Formula" (with the problem pattern as well as the solution pattern)

Examples: Emphasis on graphing. Ex 2 and all its parts, completing the square for $a = 1$, Ex 5, 9 (Ex 12 takes a long time, 15-20 minutes) Calc Ex 1 is a good mid-class break – almost no one will know how to do it until you show them.

Aloud in class: A3-6. B4, B5, B6-8, B23-30. Later: B9-18 (which are hard). Drop hints.

Section 3.3. Distance, Circles, and Ellipses.

Emphasize

1) Figure 2 and the distance formula as the Pythagorean Theorem
2) Figure 5 and the equation of a circle as the distance formula (squared)
3) scale changes (as in ellipses, Ex 7-9). This recalls lessons of 2.2.

[Ellipses are much less important than circles.]

Examples: An easy one and then 3, 4, 5, completing the square (twice) in 6.

Aloud in class: A7-14. B17-18.

Section 3.4. Graphical Factoring.

The students will have practiced basic factoring a great deal in Algebra I. Some will still make errors by not incorporating a "cross-product" term. Perhaps Figures 2 and 5 will help. (We do more sophisticated graphical factoring in Section 4.2, generalizing this approach. Here we just do quadratics.)

The emphasis should be on advanced factoring methods.

0) We have often written quadratics in this form: $ax^2 + bx + c$. Here the objective is to rewrite them in "factored form": $k(x - b)(x - c)$ [where the letters are different.]

1) The "Factor Theorem" is the reverse of the "Zero Product Rule" and how each relates to graphs (Ex 7, 8, 9).

2) how the Quadratic Theorem yields factors (Ex 7, 8)

3) do not emphasize non-integer factors and non-real factors (Ex 10-12)

For reading and writing math:

4) terminology: expression, equation (this distinction is difficult for students), factor, zero
5) state the Factor Theorem

[Stating it is very hard for them. In the equation "$f(x) = 0$" they see "x" as a solution – a particular number. However, in the problem "Factor $f(x)$" the "x" they got as a solution is now the "c" in "$x - c$" a factor. The role of "x" has changed. The factor has a general "x", not a particular "x".]

Lesson: Consider a quadratic in factored form, e.g. $(x - 3)(x + 2)$. Students should be able to identify where it crosses the x-axis before you actually graph it. Graph it. The Zero Product Rule makes it easy to see the zeros, which are the values where the graph crosses the x-axis. Now put an additional factor out front, $2(x - 3)(x + 2)$. The zeros remain unchanged. Point: The zeros can not tell you the constant factor out front.

Now, multiply out the original quadratic: $x^2 - x - 6$. Graph this. Before, the factors told you the zeros. Now, the zeros tell you the factors (except for the constant factor out front).So, the Factor Theorem reverses the information in the Zero Product Rule. If the quadratic is given in the form which is multiplied out, the graph allows you to see the zeros, which correspond to the linear factors. Pick any other point not on the x-axis and we can adjust the constant factor out front to make the graph go through that point.

Examples: Strong emphasis on graphing. Ex 5, "How much do two CDs at $14.98 each cost?" (They should be able to do this in their heads), 7, 8, 9.

Aloud in class: A1-2. A3-6, A7-8 (given them time to graph it on their calculators), A23-24, A26-28. After some practice, you can ask students to begin to factor B5-8.

Sections 3.5 and 3.6. Word Problems.

Sections 3.5 and 3.6 have very similar lessons. The difference is that the examples in Section 3.6 are be slightly more complicated.

Philosopsy: These sections could be titled "Indirect Word Problems." That is, it is about word problems in which the words and basic formulas express *operations that you do not do*. Instead, you *represent* them in algebraic notation.

The idea of the section is to get students to *notice* and *represent* the operations they would *use* to evaluate a quantity. They are represented by a formula-like algebraic expression. If they can "build their own formula" the proper equation to solve follows easily.

Many students have great difficulty with *indirect* word problems. Word problems express relationships between quantities. Often students can *evaluate* a quantity using a correct sequence of operations (when the problem is direct) and yet fail to explicitly notice the evaluation *process* (because *process* is a higher-level concept). Therefore, they often cannot *represent* the operations they use. Then, when the problem is *indirect*, they can't set up an equation to solve, and eventually they decide they are "bad at word problems."

We want to fix that. The problem is that they work comfortably with numbers as concepts, but they do not yet have operations as firm concepts. Our symbolic notation is designed to express sequences of operations (it is designed to help them "build their own formula"). So, the lessons of Chapter 1 should help here. Expressions such as "πx^2 do *not* just represent numbers and "x" is not just the numerical answer. "x" is a tool (placeholder) for representing the relevant sequence of operations (the evaluation process).

The lessons of the first two chapters should help. Students need to know how to write about operations. In a word problem, to set up the equation that will eventually need to be solved, it is critical to first know how to evaluate the quantity involved, and to know how to represent that evaluation process in symbols. This is what it means to "build your own formula." That is what "guess and check" can help (page 164, Examples 10, 12). Of course, some students can go straight to "x" without numerical examples first, but many cannot (yet).

It is important to note that in calculus obtaining the formula (functional relationship) is often more important than obtaining a numerical answer. We do not want to encourage shortcut approaches that avoid obtaining relationships (because the point here is to learn how to use algebraic notation to "Build your own formula").]

Lecture:

Outline on the board:

How to do word problems
1) Name the answer
2) Draw a picture if the problem is geometric. Label it.
3) Write down relevant formulas
4) Build your own formula (using basic formulas as building-blocks)
5) Reread the problem several times.

After a while, add in this piece of advice:

If you don't see the formula,

6) Consider a guess-and-check approach to expose the relevant operations and write those operations in symbolism.

Second lecture: Repeat the advice:

Lecture:

Outline on the board:

How to do word problems
1) Name the answer
2) Draw a picture if the problem is geometric. Label it.
3) Write down relevant formulas
4) Build your own formula (using basic formulas as building-blocks)
5) Reread the problem several times.

If you don't see the formula,

6) Consider a guess-and-check approach to expose the relevant operations and write those operations in symbolism.

Add to it:

7) Start writing! (Which is the additional advice from Section 3.6.)

Example 1: Student tickets are \$6 each. Other tickets are \$10 each. 120 tickets cost a total of \$1044. How many were student tickets?

Show how Guess-and-check can help: Guess, say, 50 were student tickets. Then compute the total cost. We are not expecting our guess to be right, we are using it to expose the operations involved so we can write them. Do not computer the results but leave the operations exposed: 50 at \$6 each makes 6(50). 120 tickets total makes 120-50 other tickets. At \$10 each they cost 10(120-50). Total cost 50(6) + (120-50)10 = 1000, not 1044.

But, that is all right. We did not expect our guess to "check". It helps us "build our own formula."

$6x + 10(120 - x)$ = the cost when x tickets are student tickets. Set this equal to 1044, and solve with inverse reverse.

Example 2: A rectangle has diagonal 20 (sketch it) and area 180. What are its sides?
Sketch and label a picture (one side is, say, "x").
Write down the area formula (mentioned in the problem). Look at the picture to see the Pythagorean Theorem is relevant. Write it down, using the letters in the picture.

$A = bh = 180$. $\quad 20^2 = b^2 + h^2$.

Two equations and two unknowns. There are several ways to solve. e.g. Find h in $bh = 180$ and plug in to the other. $20^2 = b^2 + (180/b)^2$. Then there are various ways to solve it. G&C will work. This is worth doing on the calculator. There will be some work to get the window (What is the domain? $0 < x < 20$, from the picture. It has two solutions (Why? One is the side called b and the other is the one called h.)

Another way is it multiply through by the denominator (b^2) to get a fourth-degree polynomial equation in b. Cleverly, let $x = b^2$ and it becomes a quadratic in x, which has two solutions. The take the square root (we don't need negatives) of each. Those are the two sides, which come into the work symmetrically, so b could be h and vice versa.

Example 3: Find all points in the plane that are equidistant from (3, 0) and the y- axis. That is, find all points P such that the distance from P to (3, 0) is the same as the distance from P to the y-axis.

Draw a picture. Look at it to see if you can see any such points. One is (1.5, 0) [each distance is 1½ units.] Another is (3, 3) [Each distance is 3 units.] Sketch one more point above and to the right of (3, 3). Label it P and note that label is not good enough – because we need the distance formula which requires the coordinates. So, name the answer (the general point) (x, y), which is a typical name for a point.

Use the distance formula for the distance from (x , y) to (3, 0). Now, how far is (x, y) in the first quadrant from the y-axis? x. Set $x = \sqrt{(x-3)^2 + (y-0)^2}$

Square both sides and consolidate like terms to simplify. You will get $(y^2 + 9)/6 = x$, which is a quadratic with the letters reversed from our usual. It opens to the right, instead of up.

Examples: Some emphasis on graphing, but it is not the point. Also, solving the equation they create is much less important than creating it. This section is on building formulas (which is the hard part of word problems). Ex 1 compared to 3, 5, 8, 12 with guess and check first. 9, 10 if time.

Aloud in class: A9-18. A1, B2.

Section 3.6. More on Word Problems.

In this section we are again more interested in functional relationships (formulas) than numbers. Emphasize using symbols to

Build your own formula.

Again, guess and check may help.

Experience shows that some students are defeated by word problems before they start. They don't know what to do. Our response is "Start writing!" Write what? Whatever you know and whatever basic formulas seem relevant.

The symbolic formula for an "indirect" problem is precisely parallel to the calculations use in the corresponding "direct" problem. Guess and check can help, but should eventually lead to an understanding of symbolic formulas.

Examples: Little use of graphing. Ex 1 (Note the importance of "Name the answer") , 2, 6 (this is hard for them – and it uses guess-and-check to solve), 8 (Begin with guess and check, using say, $x = 40$ which works nicely, and then go to the general formula), 9.

Aloud in class: none

Chapter 4. Powers

Presumably the students understand powers such as x^2 and x^3. They are likely to know quite a bit about integer-power monomials. But we cannot presume they know the shapes of general cubic and higher-degree polynomial graphs. So they may not be able to recognize when a graph of a polynomial is representative. Furthermore, students regularly have difficulty with fractional powers, of which the most important is the square root (Section 4.3). Polynomials are nice because they can be directly *evaluated* using only the operations of arithmetic. But *solving* polynomial equations is another matter. The only easy types to solve are: monomial, linear, quadratic, and those easily factorable (Section 4.2). Remember that guess-and-check will always work, so algebraic methods are of less significance than they used to be.

Section 4.1. Powers.

Students probably already know much of 4.1. If they do not know the identities, emphasize that properties of power functions follow from the idea of powers as repeated multiplication (even for non-integer powers).

1) powers as **repeated multiplication**, $b^2(b^3) = b^5$ [not b^6]
   [You may wish to use "x" or "2" instead of "b".]
2) zero power, 2^0, b^0
3) negative powers (like division), 2^{-p}, b^{-p}
4) powers of products $(ab)^2$, quotients of powers $(a/b)^2$
5) monomials, shapes (graphing calculator)
6) polynomials, shapes of cubics
7) local maximum ("maxima" is plural), local extrema ("extremum" is singular)
8) end-behavior
   Note: odd, even, and symmetry are not as essential as the basics.

If they do know the identities, emphasize

9) the limited number of basic shapes of monomial graphs (Fig 1, 3)
10) the possible shapes of cubics (Fig 1, 2)
11) end-behavior (which can tell us when the graph is "representative" (Ex 17), and selecting a window to show a good picture (Ex 15)
    That polynomials serve to approximate advanced functions (Ex 20) is interesting, but not central.

For reading and writing mathematics:

12) Table 4.1.6 (p. 191) with the same facts stated with different letters
    [This will be useful in the next chapter when these "power function" facts become "exponential function" facts just by changing the letters. See Table 5.1.1-3 (p. 285). Also, note that the facts hold for non-integer powers such as the square root: T4.3.6]

Examples: Strong emphasis on graphing. Powers of 2, Ex 1,2, 5, 6, 12f, 15 (very good), 17 (Figures 8&9), 19, 20.
Aloud in class: A5-8, A9-12, some factors of A13-16, A17-24, A25-26, B3, B4, B21-4, B29-32.
In-class calculator project: B5-8, B15-20

Section 4.2. Polynomial Equations.

Philosophy. Polynomial equations of degree 1 (linear equations) and degree 2 (quadratic equations) have already been thoroughly discussed in Sections 3.1 and 3.2. Therefore, the emphasis is on:

1) monomial equations, $x^p = c$, using the pth root ($1/p$ power)
2) the difference between odd and even powers (one solution if odd, two if even) [Read and write Theorem 4.2.1.]
3) guess-and-check will always work (just be sure to use a representative graph)
4) factoring nearly factored expressions (Example 8)
5) graphical factoring of cubics using the Factor Theorem
   The Factor Theorem can find factors (if it is nicely factorable) and then long division can be used to reduce the degree of the polynomial equation (Ex 9, 10). This works best for selected cubics. Graphs help (this should recall Section 3.4 and factoring quadratics). By the way, students seem to have little trouble with the long division. Do not emphasize how to long divide. Emphasize the Factor Theorem itself.
6) there are theorems that tell you where to look for factors with integer coefficients (4.2.4, 4.2.5) [Less important than the others.]

Lecture:
Outline:
How to solve polynomial equations algebraically
1) some are easy and we've already done them: lines, quadratics, monomials, examples that fit the Zero Product Rule.
2) usually we do the rest with G&C (which always works, but is not algebraic)
3) Sometimes, the rest (especially cubics) can be factored, for the ZPR.
4) If they can be factored nicely, the Factor Theorem and a graphing calculator can help as they did in 3.4 when we did only quadratics.

Example: Consider $x^2 = c$ and $x^3 = c$. Graph x^2 and x^3. Note how $x^2 = c$ may have two solutions, or zero solutions (or one), but $x^3 = c$ always has exactly one solution (it is one-to-one). Generalize to $x^{\text{even}} = c$ and $x^{\text{odd}} = c$. Solve $x^n = 17$ for $n \geq 2$ (two cases!)

Example: For "representative" graphs: Graph $x^3 + 25x^2 - 20x - 200$ in the standard window. "Trace" and change only the vertical to, say, -300 to 300. You get a graph that almost looks quadratic. It can't be representative. Ask where the rest of the "action" is – to the left or to the right?

Example "Solve $x^3 - x^2 - 5x + 6 = 0$" algebraically. It is a cubic, but factors in integers [into $(x - 2)(x^2 + x + 3) = 0$] graphically. It crosses the x-axis at the integer 2, yielding a factor of "$x - 2$". The other two zeros are not integers, but the quadratic can be found exactly by long division. By the way, random cubics probably will not factor in integers, but some do, and if they do, this technique will work.

Another, similar example, that makes a good point about checking the apparent solutions to see that they really are: Solve algebraically $x^3 - 3x^2 - 47x + 56 = 0$. We want to factor it for the ZPR. Graph the left side. It looks almost like it crosses the x-axis at -6, 1, and 8. Checking, left to right, we see that the "value" when $x = -6$ is not 0, so it does not really cross at -6. Similarly for $x = 1$. But it does check at $x = 8$. So it factors into $(x - 8)$ times something. Find the thing by long division of polynomials. Then use the QF to algebraically find the solutions associated with the quadratic.

Example. Sketch a cubic with zeros at -3, -1, and 4, and which goes through (3, -7). Find it in factored form. Students have trouble remembering the “*k*” and then how to find it.

Examples: Strong emphasis on graphing. Ex 5, 6 (representative graph), 8, 9 or 10 (show some long division), 11, 18.

Aloud in class: A3-5, B5-6

In-class projects: A7-12, B7-8, B35-37 or B29-34. Use calculators to find one factor of B9-12.

Section 4.3. Fractional Powers.

The most important non-integer powers are square roots. The distance formula uses a square root.

The emphasis should be on:

1) How to eliminate square roots. Squaring both sides of an equation can produce extraneous solutions. Squaring is dangerous! Remember to check solutions whenever that process is used (Ex 1, 5). Also, some students forget the cross-product term when squaring.
2) distance in the plane (Ex 6, 7. Ex 7 Is hard for them because they hesitate to build the two relevant formulas.)
3) solving $x^n = c$
4) solving $x^p = c$ when p is not integer and $x > 0$. Dealing with non-integer powers (Ex 10, 11) Note how the reciprocal key can be used to enter inverse powers: To solve $x^{4.7} = 1000$ use

    1000 ^ 4.7 *reciprocal enter* on a TI

    1000 ^ (1/4.7) *enter* also works, but is less elegant.

For reading and writing math:

5) State the theorem on squaring (T4.3.1, and note its use of "if..., then..." which does not assert equivalence) Distinguish between "iff" and "if...,then...".

Examples: Some graphing. Ex 1, 2, 5, 6, 7, 10 (use the reciprocal key for inverse powers), 19 or 20.

Aloud in class: A25-42, A11-24, A47-66, B25-28 (excellent), B29-38.

Section 4.4. Percents, Money, and Compounding.

In spite of the fact that percents are supposed to be an elementary topic, a great many students have difficulty with percents. Because of the real-world importance of percents, I strongly emphasize this section.

Students commonly have difficulty distinguishing between the argument and the image (Ex 11, and Ex 9 continued in Ex 15). In problems with two successive stages, they may incorrectly add when the right operation is multiplication (Ex 13, 16). In problems with averages, they may divide instead of taking roots (Ex 23, 24).

The emphasis should be on:

1) percents are used the context of **multiplication** (and division) (Ex 8, etc.) for comparison of two like quanities
2) incorporating change (immediately, so a 10% increase is thought of as "multiply by 1.10"). Note that it is often appropriate to express changes using multiplicative factors (instead of additive amounts, e.g. Ex 12-14. Stores with prices "20 off".)
3) distinguishing the argument from the image (Ex 11)

4) composition (Ex 13-18)
5) averaging in a multiplicative context (Ex 23-24) (This is hard for them. Emphasize it.)
6) money and compound interest (4.4.5 [lots of parameters!], Ex 21, 25)

Examples: Little graphing (maybe graph compound interest). Possible lecture examples: Ex 1-14, 15 (a good one), 16, 18, 20, 21, 23 (important). They need not memorize the annuity formula. Example 27 is very interesting, but not central.

Aloud in class: A5-A14, A15-24.

Section 4.5. Rational Functions.

Emphasize

1) definition of "rational function" – Note it the most complex type of function that can be evaluated with the four operations of arithmetic
2) Zero Quotient Rule (but not much emphasis)
3) locations of zeros and asymptotes (the "most interesting" places)
   [and why vertical asymptotes work like they do]
4) graphical interpretation
   display and interpret lots of graphs
5) end-behavior model and how it determines horizontal asymptotes (p. 257, box)

For reading and writing mathematics: The "difference quotient" of 4.5.3 uses functional notation.

New Examples: Do $1/(x - 1)$ [which is like $1/x$ but shifted enough to make more visible the calculator's way of dealing with vertical asymptotes] , $x/(x - 1)$, and $x^2/(x -1)$ in sequence. They are great for discussing asymptotes and end-behavior. $1/((x-3)(x-20))$ is nice for a representative graph, domain, and small image values. Use the standard window to begin with and see how little of the action is visible. Then think about the domain. Use, say, 0 to 25. Then use trace to find the images. I get a good picture with $-.1 < y < .1$.]

Examples: Strong emphasis on graphing. Ex 11-15 (mention "artifact" if there is a vertical line on the calculator graph), 16-23. Something like B13-14.

Aloud in class: A3-8, A19-21, B1, B2, B3, B4, B5-8 (9, 11, 12 are harder).

In-Class Project: A25-30

Section 4.6. Inequalities.

Some processes which are legal for solving equations do not work for solving inequalities. Errors arise easily. They can be caused by processes students apparently regard as straightforward such as "Divide both sides by x." More than anything else students need to learn that inequalities warrant different (more involved) methods. Emphasize

1) graphical solving (Ex 2, Figure 3 and Ex 4)
   [The solution to an inequality consists of x-values of points on a graph, not the points themselves. Many students do not have this straight.]
2) Solving inequalities by hand. The dangers of multiplying or dividing inequalities by negative numbers or expressions. Multiplying by negatives reverses the direction. (Ex 4-5)
3) when multiplying by "x", keeping track of whether it is positive or negative (Ex 6, Ex 7)

4) the Theorem on Zeros and Signs (4.6.3) used **graphically**.

This is an *algebraic* method if the zeros are found algebraically (even though the expression is evaluated at particular points and a calculator may be used to evaluate or look at the expression). This is a calculator-aided improvement on the old technique known as "sign-patterns" which the calculator has made obsolescent.

5) interval notation, "and" is not "or" (p. 274)

6) absolute values for expressing intervals by emphasizing the center and half-width (this is common in calculus: e.g. "$|x - a| < \delta$", Ex 13-15).

New Example: Solve $\frac{2x}{x-3} < 4$ two ways. One, use a graph. Two, use cases.

Clearly $x = 3$ is interesting, but the other interesting x-value is not so clear. You can graph and read the answer. To find the other endpoint, you may wish to use a common denominator: $\frac{2x}{x-3} - 4 < 0$ so $\frac{2x - 4(x-3)}{x-3} = \frac{-2x+12}{x-3} < 0$.

where the interest of $x = 6$ as a zero of the top is clear and the application of the Theorem on Zeros and Signs is clear.

New Example: Solve $|x - 5| < 2$ to get $3 < x < 7$. Interpret the former as all the points (on a line – not in the plane) within 2 units of 5. Ask them to solve something similar, say

$|x - 4| < 3$, in their heads. Then show them how to get from $1 < x < 7$ back to $|x - 4| < 3$ by noticing the center and half-with of the interval. Picture the center and half-width on the board. Do Theorem 4.6.6. and then have them convert $7 < x < 11$ to "$|x - c| < d$" form.

Then, do some with non-integer centers, eg. $17 < x < 20$ and $4.5 < x < 4.9$.

You might remark as you go along that, in calculus, we often prefer the "$|x - c| < d$" form because it emphasizes the center and is appropriate for expressing the idea of "close" to some point.

Examples: Strong emphasis on graphing. Ex 2, 4, 5, 6, 7 (6 and 7 are critical), 8, 9, 13-15.

Aloud in class: A2-4, and (harder) A17-19, A36-43.

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Chapter 5. Exponential and Logarithmic Functions

Most precalculus students do not have a good grasp of exponential functions. Even fewer understand logarithmic functions. Furthermore, the use of "*e*" is hard to justify without delving into calculus. But, at least exponentials can be connected to repeated multiplication, so we start there. Unfortunately, logarithmic functions are connected to easier ideas only through the concept of inverses. That is, logs are good for solving the equation $b^x = c$.

This is a chapter where we can not assume the students know the basics. Even facts such as "10^3 = one thousand" are not necessarily common knowledge.

Section 5.1. Exponents and Logarithms.

Note the parallels between power functions and exponential functions. Emphasize

1) the basics (10^x, first for various integer values of x. Then $x = 1/2$.) (Relating this to "scientific notation," which calculators use for large numbers, may help, but is not necessary)
2) the correspondence of power and exponential functions (Table 5.1.1-3)
[b^p can be x^p (power function) or b^x (exponential function). Recall properties of powers and rewrite them with letters emphasizing exponentials. [If you use base 10 (instead of b), the switch to base 2 or b or e is easy later.]
3) the correspondence of exponential and logarithmic forms (5.1.4)
[This defines logs. "**logs are exponents**"]
4) properties of exponentials and logs come in pairs (5.1.6)
[Derive log properties by inspection (not lots of steps) of exponential properties. logs will not become "real" to students until they can see them as the exponents they are.]
5) 5.1.3L, "$\log(c^x) = x \log c$" which is the most useful log fact. It helps solve for unknown exponents.

Emphasize solving for unknown exponents (problems like A19-A28 and B3-B10. A17-18 use roots, not logs).

For reading and writing mathematics:

6) Tables 4.1.6 and 5.1.4 have the same facts stated twice, with different letters for different emphasis. Note how facts about "power functions" are also "exponential function" facts when "*x*" is in a different position. Note how properties of logs follow by inspection. I ask the students to supply the basic power-function facts and then I have then restate them for exponential functions.

Examples: Only some graphing. Calculator Exercise 1 (p. 285), Calculator Exercise 2 (p. 287), Show the growth of 10^x and even 2^x. Comment on how fast they grow. Show that 10^x and $\log x$ have the "mirror-image through the line $y = x$" property from Section 2.3. Most examples are necessary. Ex 4-15, 17, 18.

Emphasize solving for unknown exponents.

In-class project: A short writing project: Calculator Exercise 5. Can they write what they discover? Answer: $\log(10x) = (\log x) + 1$.

[or] $\log(10^n x) = (\log x) + n$.

Aloud in class: A37-48, A12-16, 29-34, A35, A5-7.

Section 5.2. Base 2 and Base *e*.

There are many important mathematical models that fit under the general "exponential model" umbrella. Emphasize

1) powers of 2 (Begin with these. They are simple and important in computers)
2) doubling-time model (regarded as a scale change of 2^t, Ex 1-3)
   (scale changes from Section 2.2 play a big role)
   Memorize Figure 2.
   2A) What is a mathematical "model"? (A math formula intended to quantify some real-world phenomenon)
3) half-life model (regarded as a scale change of $(1/2)^t$, Ex 4-6)
   [Recall scale changes from Section 2.2.]
   Memorize Figure 4.
4) compound-interest model (Ex 7-9)
5) continuous compounding
   [The importance of "*e*" is addressed, but will not be clear until calculus. There is a partial answer to "Why *e*?" is section 5.4. You need not try hard to explain the significance of *e* except to say "It is important in calculus."]
6) exponential model (which is one general model which handles all the others) [Since this is the usual calculus model, if your students are heading to calculus they should see this, especially in comparison to the models we did with base 2 and 1/2. Note how "time $t = 0$" can be chosen to be any convenient time.]

Examples: Only some graphing. There are many good examples. Ex 1 and scale changes, 3, 7, 9, Ex 15. Examples 16ff with change of base for exponentials are interesting, but be sure to cover the rest first. Change-of-base for logarithms is not a major topic. Do not spend your time on it.

Aloud in class: A13-22. Later A27-31. They should be able to do B3-4 aloud.

Section 5.3. Applications.

There are many interesting applications of logarithmic scales (more than we can cover in the time we spend). The applications are ones in which multiplicative factors are more relevant than additive amounts. They should find the examples about the growth of money interesting, and most students are not familiar with the idea of non-uniform graphical scales (Figure 2).

The emphasis should be on

1) change-of-variable as in the Richter scale (Ex 1) and decibels (Ex 2)
   (Everyone can relate to the loudness of sound, and everyone hears about big earthquakes. These are excellent real-world topics.)
2) the value of multiplicative scales (as opposed to the usual additive scales) Ex 3, 4
3) graphs with logarithmic scales (especially those with money) Fig 2
   (Money in important. Graphs to depict money growth must use semi-log scales – the usual scale misrepresents growth!)
4) exponential model becomes linear with a change-of-variable by taking logs (that is, a **constant** growth **rate** gives a picture with **constant** slope, whereas the usual scales would yield increasing slope for a constant growth rate.)

Examples: Little graphing. Ex 1-4 (especially 4), and 6.

Aloud in class: none.

Section 5.4. More Applications. This section could be skipped to save time, but does have the interesting "power model."

Emphasize

1) The power model becomes a linear model with a change-of-variable.

The principle of the slide rule is interesting (B5), and a slide rule is a good lecture prop, but this knowledge is not essential any more. Graphing may be used to illustrate the effect of changing variables.

The partial answer to “Why *e*?” is interesting, but not critical at this time.

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Chapter 6. Trigonometry

This chapter emphasizes solving triangles. Chapter 7 emphasizes trig for calculus.

Both the right-triangle and unit-circle definitions of the trigonometric functions are emphasized immediately (Section 6.2). The unit-circle interpretation explains a lot that the right-triangle interpretation can not.

Most students will have seen some trig, and many will have taken a course in trig. Clearly they have an advantage over those who have not seen any trig, and it is fair to announce that those who have not studied trig will need to work even harder in this chapter. But, having had trig is not as great an advantage as might be expected. Many did not memorize anything and don't even recall that "sine is opposite over hypotenuse." Having "had" a subject is not much of an advantage if you don't remember anything!

It turns out that including the unit-circle definitions right away makes some things much easier, so students who have not had trig can catch up fast. In particular, the unit-circle approach is best for understanding trig functions of angles that do not fit in right triangles (e.g. obtuse angles) and for understanding inverse trig functions.

When solving a triangle, the emphasis of the text is on developing a plan. That is, the emphasis is on recognizing which results to use and how to use them – and recognizing whether the calculation will yield an ambiguous answer. Some trig calculations yield two solutions instead of only one because the geometric case is ambiguous. The ambiguous calculations are: solving for an angle in the Law of Sines (Angle-Side-Side), and solving for an adjacent side in the Law of Cosines.

Section 6.1 reminds students of the geometric cases that determine a triangle, or almost determine a triangle. I hope they learned the compass and straightedge constructions in Geometry so they can tell what parts do not determine a triangle, but the constructions are not required here. The Pythagorean Theorem and the sum of the angles in a triangle are given. Do not spend much time on 6.1. **Emphasize the ambiguity of Angle-Side-Side**.

Section 6.2 defines sine, cosine, and tangent on the unit circle and also interprets them in right triangles. Inverse functions are also discussed. **Solve some right triangles**. Have them **memorize both the right-triangle and unit-circle definitions**. It is a long section.

Section 6.3 gives the Law of Sines and the Law of Cosines for solving non-right triangles. The cases to which they apply are carefully considered and examples of what can go wrong are discussed. **Emphasize how to decide which law to use and emphasize when the Law of Sines yields an ambiguous answer**.

Section 6.4 emphasizes planning how to solve a geometric figure. It also introduces some applications. **Emphasize planning how to find the solution**.

Here are the numerical tools of trigonometry:

1) The definitions of the trig functions (6.2.2, and especially 6.2.4)
2) The sum of the angles in a triangle is 180° (6.1.4).
3) The Pythagorean Theorem (6.1.8).
4) The Law of Sines (6.3.3).
5) The Law of Cosines (6.3.4).

These are all the basic numerical tools of trigonometry. When students understand these, they can efficiently solve any solvable triangle.

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Section 6.1. Geometry for Trigonometry.

Do not spend much time on this section. The one key point is that **Angle-Side-Side may not yield a unique triangle**. (This will come up frequently when using the Law of Sines.)

Triangles have 6 parts (3 sides and 3 angles). Whether or not a given three parts determine the other three is discussed in geometry. Section 6.1 describes the various cases. When we use the Law of Sines or the Law of Cosines to solve a triangle (coming up in Section 6.3), certain examples yield ambiguous results. Of course, ambiguous numerical results arise in precisely the ambiguous geometric cases. So Section 6.1 looks forward to those laws with an eye toward when they give a single answer and when they yield two solutions (Solving for an angle in the Law of Sines and solving for an adjacent side in the Law of Cosines yield 2 solutions).

In class we do not need to emphasize the geometric cases which give unique triangles (SSS, SAS, ASA). Students expect unique results. It is the non-unique results they need to worry about.

The emphasis should be on

1) 6.1.4 – two angles yield the third
   which explains why AAS is treated as ASA. AAS is the most difficult geometric construction (don't explain it in class – unless you have a lot of time to spend on it!), but it is easy to solve the triangle trigonometrically, because the third angle is obtained by subtracting the two given angles from 180°.
2) ASS (the most dangerous and interesting case in trigonometry).
   I write "ASS" because that is the usual order in which laid out, left-to-right, for construction. Some call it "SSA" to avoid the writing the "dirty word." But I tell my class that case **is** a dirty word because there is a trick to solving such dangerous triangles. If you dislike the abbreviation ASS, perhaps A-S-S is a bit better. Or, you can simply write out the whole thing "Angle-Side-Side."
   When solving for an unknown angle in the Law of Sines, equations such as "$\sin x = c$" arise. They have **two** solutions for the same reason that ASS produces **two** solutions. Solving, for example, "$\sin x = 1/2$," your calculator yields one solution (30°) – you have to think of the other (180° - 30°= 150°) yourself. Many students forget to think of the second solution (Figures 3-4, 16, 18).
3) recognizing the various cases
4) The angles of a triangle sum to 180 degrees.
5) The Pythagorean Theorem (which they already know)

I hope students know how to do geometric constructions. They are quite illuminating. However, if they don't know how to do them, I do not require them to learn them in this course. That belongs to a different course.

Examples: No graphing. Use a compass to illustrate the cases, especially ASS. Possible lecture examples: "similar" (Figure 2), Figures 3 and 4, SSS, SAS, Ex 4, 5, 10-13.

You can make a handy compass by simply tying a string to a piece of chalk. Hold the string down at the chosen radius with one finger or thumb on one hand and make an arc with the chalk in the other hand. I do not require compass and straightedge constructions from students. Good freehand sketches are enough, although constructions are even better.

Section 6.2. Trigonometric Functions.

This is a long section (2-3 days). It appears to have a great deal of material, but, in fact, it is easy to cover in 2 days. My students have had an easy time with it. Furthermore, I have found that studying it all at once works very well. Actually, everything is so interconnected that it can all be covered once in a single day's lecture and each part reinforces each other part. The concepts they do not totally grasp in two days (for example, the answer to "sin $x = c$" is not unique, or that "tangent is opposite over adjacent" applies only in *right* triangles) reappear frequently in upcoming material and they learn them there if not here.

In the end, the students will have to understand all trig material as a unified whole. Breaking it up into distinct parts does not work as well as treating it all at once.

Of course, **students must memorize "Sine is opposite over hypotenuse", "cosine is adjacent over hypotenuse," and "tangent is opposite over adjacent" for right triangles**. But, that is not enough.

Do **not** let the students be satisfied with a right-triangle interpretation of sine, cosine, and tangent. Also **emphasize the unit-circle definitions of trig functions and the unit-circle interpretations of inverse trig functions**.

The lecture should be on

1) definitions of sine, cosine, and tangent (**Memorize** 6.2.2, Figures 4-6)
2) solving some right triangles for sides given the angle (not too many – we do not want to leave the impression that trig is only a right-triangle subject)
3) unit circles
    a) trig functions (**Memorize** Figures 19 and 39)
        Sine is the vertical coordinate, cosine is the horizontal coordinate
    b) how the graph of sine is generated (Figure 24)
    c) inverse trig functions
    ["$\sin^{-1} x$" can be read aloud "the angle whose sign is x," or "arcsine x" or "inverse sine x." The first is most illuminating.]
4) identify angles from their sine values (Figures 25 and 30)
    [Note that the sine of an angle does not determine the angle's quadrant. You can mention "reference angles" if you want, but they are not covered thoroughly until 7.2.]
5) solve some right triangles for the angle given two sides
6) splitting isosceles triangles into two right triangles (Ex 4, 5)
7) **memorize the two most important triangles** with easy angles and trig function values (the 45° right triangle, and the 30°-60° right triangle, Figures 46 and 47)

Examples: Some graphing, not necessarily with the calculator. Be sure to draw lots of unit circles and estimate sine, cosine, arcsine, and arccosine from the picture. This brings reality to the unit-circle definitions. All they examples are good. Just begin at the beginning. Use your own right-triangle trig examples paralleling the other examples in the text.

For the second or third day, I love homework B68 and B69: The width of my thumb subtends 2° at arm's length. The span of my outspread fingers subtends about 20° at arm's length. Show how to find out the numbers 2° and 20° and see if it is also true for you and them, even though they may be taller or shorter.

Aloud in class: B5-8 (you can sketch your own), A67-70, A25-41.

Section 6.3. Solving Triangles.

Prove the Law of Sines and note how it can be ambiguous when used to solve for an angle (the geometric case is then ASS).

Begin in the text's order:

1) Derive the SAS Area Formula (6.3.1) as a nice easy use of trig, but is not the main theme.
2) Derive the Law of Sines. (It is also a nice use of trig.)
3) Use the Law of Sines on non-right triangles. Solve for sides first (AAS or

ASA), because the solution for a side is not ambiguous. Then be sure to include some ASS examples where the solution is really the obtuse angle. You can make up great examples as you go along in class. Take a ruler and protractor. Use the ruler to draw triangles on the board. Measure three parts and use trig to derive the others. Then measure them to see how close the answers were. On the ASS cases, note the **two** triangles that would fit the given parts, even if only one was actually drawn. Using a compass (of string) here can help to show how there could be a different side of the same length.

4) State the Law of Cosines. Note how close it is to the Pythagorean Theorem. [You may or may not decide to derive it the first day. I do derive it, but usually not until after using it on several examples – maybe the second day.]
5) Use the Law of Cosines on SAS and SSS examples. Note how obtuse angles are no problem (unlike in the Law of Sines). Again, ruler and protractor examples work well.
6) Treat the straightforward cases first and foremost. But, later show how using the Law of Cosines on ASS is possible. Note how it yields a Quadratic equation, which has an ambiguous solution precisely when the geometry is ASS and ambiguous, and when the Law of Sines produces an ambiguous solution.] Sketch an ASS case. Measure the two sides and the angle. Solve it and show how the two solutions correspond to the geometry.

The emphasis should be on
1) the Law of Sines and the Law of Cosines
2) when to use which result (which is also a big part of the next section)
3) the geometric reason a case can be ambiguous
4) The Law of Sines is ambiguous when solving for an unknown angle. "sin $x = c$" has **two** important solutions. The Law of Sines is not ambiguous when used to solve for an unknown side.
5) The Law of Cosines is not ambiguous when solving for the opposite side or the angle (but it can be ambiguous when solving for an adjacent side). It generalizes the Pythagorean Theorem.

Have students program their calculators with the Law of Sines two ways. One to solve for the far side in S-A-S and one to solve for the included angle in S-S-S.

Examples: No graphing, some computing. Lots of triangle drawing. All examples are good, but it is easy to make up very similar examples, leaving the textbook examples for the students to read.

Aloud in class: B3-10.

Section 6.4. Solving Figures. This section emphasizes planning how to solve a triangle or more complicated figure. Emphasize the *plan*. Do not worry too much about the numbers obtained by implementing the plan. (Students are good at plugging in to formulas and tend to have no trouble getting the numbers, if they have a plan.)

The emphasis should be on
1) plans for solving triangles given three parts (as in Figures 3-7)
2) plans for solving more complicated figures (begin with one like Figure 8. Again, you can use a ruler and protractor to set up you own examples in class.)
3) expressing direction on the surface of the earth ("bearing") and drawing appropriate trigonometric figures. (Begin with basic examples like Figures 14 and 15 and work up to two-leg trips like Figure 16.)

Examples: No graphing, possibly lots of calculating (if you do the work and don't just create the plan). Random examples like 3-7. Use different numbers, but do examples similar to 8, 9, 11, 13 (two ways), 14.

Aloud in class: B1-3 (later, B29ff – sketch them on the board and label the vertices so the students can communicate).

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Chapter 7. Trigonometry for Calculus

Calculus uses trig identities and angles measured in radians. Section 7.1 introduces radians and Sections 7.2 and 7.3 have the trig identities that are actually useful in calculus. In this text trig identities are not treated as arbitrary exercises. Each one is actually useful for increasing understanding. The approach to trig identities and inverse trig facts is visual – based on the unit circle and sometimes right triangles – as far as possible. Of course, the more complicated identities (such as the sine of a sum of angles) do not have simple pictures. Section 7.4 discusses some entertaining facts about waves.

This chapter focuses on **deduction**. Students should learn how to take things they know and combine them to create new results. In 7.1 there are formulas for area and arc length that need not be memorized because they are easy to derive. In 7.2 there are many simple trig identities--more than they need to memorize because students can learn how to derive them. The more-complicated trig identities in 7.3 serve to teach students something about how deduction works.

Section 7.1. Arc Length and Radians.

This section emphasizes arc length and radian measure for angles. The lecture is pretty straightforward, but radians seem to students like a new version of something they already know well enough, so it is not natural or necessary for students to switch their thinking to radians until calculus. Expect students to think in degrees even after studying this section unless you stress radians very hard.

Emphasize

1) Definition of "radian" (7.1.1) and mental images of it (Figure 2). Sketches of angles and their radian measure (Figures 3-5).
2) Arc length and area of a sector **derived** as proportions (above 7.1.4 and 7.1.5) **Memorize** the derivation and pictures (Figures 3 and 8) from which the arc length and area formulas are derived. This is an important and illuminating use of proportionality.
3) Lesser emphasis: the unit-circle idea that sin x is near x for small x (Figures 9 and 10). (tan x is also near x for small x.)
4) Unit conversion by multiplying by 1. Example 4 and HW A25-26, where (60 minutes)/(1 degree) = 1.

Examples: Little graphing (Fig 10). Some geometric pictures (Figures 2, 13, 8, 9) [Take a string compass]. Unit conversion by multiplying by 1 [e.g. "Convert 3°26' to decimal degrees."] Examples like 1-3, 4-5, derive area of a section and arc length in degrees from proportions, a problem like B7, say with AB = 20 and OA = 30.

Aloud in class: None. Possibly A1-4, A29-36.

Section 7.2. Identities.

This section emphasizes the trig identities used in calculus that can be derived with a picture, either a unit-circle picture or a right-triangle picture. Section 7.3 derives additional identities that are more complicated. Emphasize

1) **Good, illuminating, sketches, reasonably large, and well-labeled**
2) memorize Figure 1
3) reference angles (and how two angles with the same reference angle have the same trig function values, except possibly for a plus or minus sign, Figures 2 and 3).

4) orientation affects signs (Figure 3) in trig, unlike in geometry, where is does not
5) sketching angles related to angle θ, such as angle θ+π or θ+π/2 (Figure 8)
[Even sketching them seems to cause students difficulties – but the real goal is next.]
6) reading unit-circle pictures to derive trig identities (Figure 8). Emphasize labeling all the parts correctly.
7) identities for angles with the same reference angles, or that differ by 180°or by 90°.
8) how right-triangle pictures can be used to derive trig identities (Figures 14-19) [This technique comes up in calculus.] Illustrating, say, "sin x = c" [with all labels] and "$\tan^{-1} x = \theta$."
9) complete and useful **labeling** (You can't stress this too much)
10) definitions of secant, cosecant, and cotangent (7.2.11, but not much else about them except 7.2.13)
11) Do some examples in radians and some in degrees. This will help students become familiar with radians.

We may categorize trig identities into three types:
1) Identities that follow easily from the unit circle picture (they often have π or π/2 or 180° or 90° in them) and that are related to reference angles (e.g. Examples1-6, 7-8).
2) Results that can be easily derived from a right-triangle picture (such as Figures 16-19).
3) The more-difficult identities in 7.3 that take quite a bit of work to derive and which are hard to memorize. The latter ones follow from the sum identities.

Students should be held responsible for the methods from which types 1 and 2 follow easily. Calculus students will need type 3, but can look them up when they need them, if they are aware they exist. I emphasize that there are such formulas, but do not require them to be memorized. However, I do require them to learn the derivation of each of 7.3.1 through 7.3.5, given either the initial picture (for 7.3.1) or the previous result (for each of 7.3.2-5).

Examples from 7.2. Little or no graphing. Many geometric pictures. Examples 1-6, Examples 7 and 8 are important for introducing the unit-circle type of picture. Examples 11-12, 13, 14-18.

Aloud in class: A3-8, A9-12, A33-34, A35-38. With a calculator: A21-26, 27-32.

Section 7.3. More Identities.

All the identities in this section follow from the sum-of-angles identities. The derivations will help students learn trig.

This section is used to help students learn to follow a line of reasoning. Many students want to treat math as a set of one-step algorithms and find problems with several steps very difficult. The derivations in this section can be used to help them realize how one result can follow from another.

Emphasize
1) The derivations of all the identities 7.3.1-5, but especially the steps illustrated in Figures 1-3 that lead to 7.3.1A and B.
2) awareness that there are
a) sum and difference identities (7.3.1). The proof (Figures 1-3) is an excellent example of using trig function definitions.
b) difference identities (7.3.2) [Put "-β" in the place of "β" and simplify]

c) double-angle identities (7.3.3) [The proof is a simple use of placeholders]
d) identities for sine squared and cosine squared in terms of cosine of the double angle (7.3.4 is important in calculus)
e) half-angle identities (7.3.5) [A clever use of placeholders: Put "θ/2" in the place of "θ".]
[Note how each identity follows from the previous identities.]

I, personally, do not require the identities to be memorized, but I do require the students to be able to derive any of these results given the previous key result from which it follows in a few steps. Studying how these all fit together helps students learn trig and learn about connections in mathematics, almost like an introduction to proof.

Lecture: All of the proofs of 7.3.1A and B (students can contribute, if led properly). The proof of 7.3.2A. (Say B is similar, and ask them to provide the steps in class.) 7.3.3A and B, followed by C and they can help with D. 7.3.4 comes from 7.3.3 easily. 7.3.5A is clever, but 7.3.5B is similar.

Aloud in class: They can help construct the proofs, if led properly. Homework A1-2. They could be asked to "Give the equation" for A11-16, but these are not trivial.

Section 7.4. Waves.

This section is short and fun. It relates waves, period, frequency, and wavelength. The idea of waves canceling each other is interesting and important (Example 11, Figure 8). Students will see this idea in physics when they study light. The frequencies and wavelengths of radio and light waves will surprise some students and the speed of sound is interesting.

Examples: Strong emphasis on graphing. Examples 4-8, 11, 12.