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Language Concepts of Mathematics

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Attention has been focused on parallels between teaching mathematics and teaching second languages by Borosi and Agor (1990). The parallels can indeed be very close.

Mathematical results are expressed in a foreign language. That language, like other languages, has its own grammar, syntax, vocabulary, word order, synonyms, negations, conventions, abbreviations, sentence structure, and paragraph structure (Esty, 1991). I call the language Mathematics (spelled with a capital letter, like other languages). Mathematics has certain features unparalleled in other languages, such as *representation* (for example, theorems expressed with "x" also apply to "b" and to "2x - 5").

The language of mathematics is both a means of communication and an instrument of thought (Kaput, 1988, p.167). The purpose of this article is to describe essential language concepts which have been underemphasized in the usual mathematics curriculum and to discuss some of the basic patterns of mathematical expression and thought. The ideas are of particular relevance to the grades 9 through 12 curriculum standards of the National Council of Teachers of Mathematics (1989): Standard 2, "Mathematics as Communication," Standard 3, "Mathematics as Reasoning," and Standard 14, "Mathematical Structure." The ideas are applicable to college instruction as well because concepts are better learned late than never. This article demonstrates the importance of language concepts in the interpretation of mathematical results and suggests ways to improve the reading comprehension, writing ability, and reasoning skills of school and college students by integrating language topics into the curriculum. These topics can be effectively taught to a wide variety of students, including "math anxious" students as well as strong students (Esty & Teppo, in press).

Language shapes thought. We are likely to think the kinds of thoughts that our language makes convenient to express. The language of mathematics not only facilitates expression of mathematical thoughts, including "those modes of thought that are essentially algebraic" (Love, 1986, p.49), it incorporates essential mathematical concepts. Furthermore, it need be learned only once and is then good forever after. Fluency in it provides access to the whole world of mathematics.

Mathematics as a Language

We can be completely serious when we assert that Mathematics is a foreign language. For example, letters are used as variables in sentences in several different ways which are unique to Mathematics (Usiskin, 1988). These usages can be successfully taught (Kieran, 1989, p.42). Another language aspect is that many expressions require conventional interpretations. For example, "$3 + 2x^2$" is interpreted according to the algebraic conventions and not simply left-to-right. Among its many foreign features is one which is relatively simple -- it has repetitive patterns of expression which can be described by a very limited number of elementary results from logic. These results can be extracted, condensed, and successfully taught (Esty & Teppo, in press).

No one doubts that logic pervades the *development* of mathematics, but its importance for the *expression* and *concepts* of mathematics has been underestimated. Reading comprehension and writing skills require the obvious reasoning component of logic, but there is far more to logic than just reasoning. For example, the various kinds of mathematical sentences (open sentences, generalizations, existence sentences) are distinguished in logic and the different types of uses of letters as variables are studied in logic. In logic, the unifying logical concept of *generalization* allows the essential similarities of *many* examples to be regarded apart as *one* new thing at a more abstract conceptual level. This mental process is fundamental to many basic mathematical conceptions such as *function* and *set*. I will argue that these aspects of the language of mathematics are essential to the understanding of algebra, sets, geometry, and all other areas of secondary-school mathematics, as well as the higher-level subject areas such as calculus. Kieran (1989, p.39) notes, "this particular aspect of algebra appears to be one that never really does get sorted out by most students throughout their entire high school algebra career." These observations can be combined to suggest a supplemental approach to mathematics education. It is to integrate study of the *language itself* into the curriculum.

Many students do seem to learn enough of the language of mathematics to succeed at algebra. But many get good grades and still cannot get through the first few weeks of calculus. An alarming 54% of first year college calculus students do not complete the first year with a grade of D or better (Anderson & Loftsgarden, 1987), and these students are not the ones who did poorly at high-school algebra. Every year about half the students in secondary-school mathematics get the message they are "not good at math" and do not continue with the next mathematics course (Committee on the Mathematical Sciences, 1990, p.59). We should *not* deduce that all those students are "just not good enough." I submit that our current approach has failed to provide them the language skills they need. The fact that the majority of students do not master the basic mathematical language concepts tells us that we, as educators, should rethink how we teach these concepts. It is a telling point that few of the language concepts in this article receive significant attention in any school textbook series.

Language Concepts in Algebraic Sentences

It is useful to categorize language concepts in two ways -- according to whether their primary application is to individual mathematical sentences (equations, identities, theorems) or to paragraphs (that is, equation-solving and proofs). I begin with a discussion of language topics which apply to sentences. Here is the way a mathematician might write an important theorem on the chalkboard.

Example 1: Zero Product Rule: $bc = 0$ if and only if $b = 0$ or $c = 0$.

This theorem gives the primary justification for factoring. It tells us one way to solve equations. For example, it tells us that $(x - 3)(x + 5) = 0$ if and only if $x - 3 = 0$ or $x + 5 = 0$. Then, employing Uniqueness of Addition, "$b = c$ if and only if $b + d = c + d$," we see that the equation is true if and only if $x = 3$ or $x = -5$.

The Zero Product Rule is a statement which uses three sophisticated language concepts: representation (letters represent expressions which may use other letters, that is, letters serve as placeholders), generalization, and logical connectives.

Mathematicians know that, in Example 1, "b" and "c" are variables which may represent any real numbers; the use of the particular letters "b" and "c" is not essential. Mathematicians know that there is an implicit "For all real numbers b and c" at the beginning of that theorem, which would be explicit in a text but which is commonly omitted on chalkboards. And mathematicians know that "if and only if" means that the second sentence, "$b = 0$ or $c = 0$," is "equivalent" to and may (and often should) replace the first, "bc = 0." The language lessons of these observations are discussed next.

Representation and Generalization. While studying generalizations students should learn that theorems which apply to *all* values (from a specified set) apply to any values (from that set), even if their names are unusual. Thus "bc = 0 if and only if b = 0 or c = 0" applies to the equation in Example 1, "(x - 3)(x + 5) = 0," because "x - 3" and "x + 5" represent real numbers, and "b" can be any real number.

How would these lessons be taught in a language class? When theorems apply to equations, before assigning problems which merely seek the answer, we could use a few problems to ask students explicitly, "What is the "b" of the theorem?" Also, students would be required to rewrite that theorem and others like it using different letters (different variables). "xy = 0 if and only if x = 0 or y = 0." "f(x)g(x) = 0 if and only if f(x) = 0 or g(x) = 0." Quantified variables can be replaced throughout a sentence with other letters. Homework and exam problems should emphasize such letter-switching for its own sake as a language skill. If all our problems appear to emphasize finding solutions the students quickly learn that only solutions are important -- the valuable language lessons are lost.

In a language course students would be required to translate into Mathematics simple sentences such as "adding a number to both sides of an inequality yields an equivalent inequality in the same direction." "b < c if and only if b + d < c + d." Almost every freshman taking college mathematics understands this idea, but I have found that a substantial majority cannot state it properly using variables (that is, in Mathematics) until they undergo additional training which emphasizes *how* algebra expresses its ideas (as opposed to *what* the content of the idea is). We expect students to read theorems in Mathematics, but we rarely ask them to write them. I have the distinct impression that instructors are afraid to ask students to state theorems -- because they know that the responses would, on the average, be terrible. This is tantamount to an admission that we have not taught students to express themselves in Mathematics. Is it any surprise that they cannot read it with full comprehension? Homework problems which require students to practice stating methods and results "in algebra" can help them master the difficult concept of representation and would be appropriate if spread throughout any mathematics course.

Representation, Functions, and Word Order. The mathematical method of describing functions requires the concept of representation. Weeks after functional notation has been introduced, if an instructor defines a function f by $f(x) = x^2$ (for all x) and then asks students "What is f(3)?" most students will correctly reply, "9" (Herscovics, 1989, p.75). But the questions "What is "f(y)?" and "What is f(x + h)?" elicit very poor responses -- the fundamental lessons of representation in generalizations have not been learned. Even the idea that "f" (as opposed to "f(x)") is defined by "$f(x) = x^2$" is usually unclear. This is well-known to be a difficult concept (Markovits, Eylon, and Bruckheimer, 1988) and many first year algebra texts avoid it (and thus the key idea of *representation* and placeholders) in favor of the less conceptual "$y = x^2$" notation.

Class days devoted to the subject of word order would be well-spent. When the time comes for students to evaluate f(x + h) they would be more capable of recognizing that "x + h", which represents a number, comes *first* and *then* f is applied. Students often apply f to "x" and not to "x + h," which demonstrates that they do not fully grasp that the definition of f in terms of variable "x" is a generalization in which the name of the variable is irrelevant. Note that our standard functional notation, "f(x)", may cause problems because it is not evaluated left-to-right like English. "x" is first, then "f" is applied.

Conventions about order are fundamental to interpretation of mathematical expressions, but many students do not know even the standard algebraic conventions. Of course, they *should* know them. They have been told them. They read sentences employing them in almost every mathematics lesson. Nevertheless, too many students do not know them. How can the language approach help students learn the importance of order?

Language texts discuss syntax, that is, the proper arrangement of the components of sentences. A language course would explain the conventions about the use of parentheses and ask the students to identify the order in which the operations are executed in numerous expressions. For example, "x + 5x" is not equivalent to "(x + 5)x" and the order in which the expression "$2 + 3x^2$" is evaluated is certainly not left-to-right. Students would evaluate expressions like these where the point is to recognize the importance of order, not to solve an equation. If we want to make a point, our homework must occasionally *emphasize* the point, not merely utilize it. Then the questions would appear on an exam, "What are the algebraic conventions about order?" and, "For each convention give an example of an expression in which the order of operations is *not* left-to-right."

Study of the conventional rules in algebra is good preparation for the study of functions, since functions are also rules. When students grasp the rules of order for evaluating expressions such as "$2 + 3x^2$" in which 3 multiplies x^2 (not "x") and then 2 is added to the result (not to "x"), they are a step closer to understanding the expressions "$f(x^2)$" and "$f(x + h)$" which also have particular orders.

There is another happy consequence of emphasizing order. If students recognize the order in which an expression must be evaluated they can easily learn to solve equations in which the key idea is to do the inverse operations in the reverse order (also known as "doing and undoing"). This is an algebraic procedure in which "you must think precisely the opposite of the way you would solve it using arithmetic" (Usiskin, 1988, p.13). This concept can be mastered by very young students even before they take algebra.

Example 2: When the problem is, "Solve $3(x + 5) = 21$," the particular value "21" is not the key to the solution process. The order of operations in the expression "$3(x + 5)$" is. The original order, beginning with x, is to add 5 and then multiply by 3. To solve it, do the inverse operations in the reverse order. First divide by 3 and then subtract 5.

Students need to learn about word order, not only for this "inverse-reverse" equation-solving method, but also to understand identities and other equation-solving methods. For example, The Zero Product Rule (Example 1) is often employed after factoring, and the purpose of factoring is to change the order of the operations in the expression so that multiplication is expressed *last*.

Order is also a key to understanding simplification, which many students find to be an unmotivated process (Davis, 1989, p.117). The purpose of simplification of an algebraic expression is not to change the numbers, but to change the sequence of operations used to express the numbers (to a preferable sequence). Preferred sequences can be expressed as algebraic patterns such as the Zero Product pattern ($ab = 0$) or the "inverse-reverse" pattern ($f(x) = c$) in which we want $f(x)$ expressed with only one appearance of the unknown "x" (Example 2). Order of operations is a necessary preliminary concept for simplification and emphasis on which orders are preferable can be used to motivate simplification.

Generalizations and Open Sentences. Students must learn to recognize generalizations and distinguish them from open sentences. The main problem is that generalizations are often abbreviated, in which case they have the same *appearance* as open sentences, although their *interpretation* is much different. In the English sentence, "He set the chess set down," the word "set" is used with two distinct meanings, yet, in context, there is no difficulty distinguishing which meaning is which. But consider the difficulties confronting an algebra student. Here are three sentences in Mathematics which have apparent similarities but different interpretations:

$(x + 1)^2 = 5$;

$(x + 1)^2 = x^2 + 2x + 1$;

Let $f(x) = (x + 1)^2$.

Anyone fluent in Mathematics understands that "$(x + 1)^2 = 5$" is an equation which is true for some values of x and false for others. It is a sentence about x. The knowledgeable reader expects it to be an equation to be solved.

The second sentence, "$(x + 1)^2 = x^2 + 2x + 1$," is an identity, true for all values of x. It is not about x. An implicit "for all x" has been suppressed, as is quite common. One side of the equation may be used to replace the other in any sentence. Therefore it can play a role in the process of solving an equation. The informed reader may use this equation, but does not expect to solve it.

The third sentence, "Let $f(x) = (x + 1)^2$" is also true, for a different reason. It is a definition of the function f, and is therefore true "by definition." Moreover, the sentence will probably remain true only until the next problem when there will be a new, different, f.

In algebra the ability to determine meaning by appearance requires a certain amount of fluency in Mathematics. These three cases are distinguished and studied in the subject of logic.

There are only three basic types of mathematical sentences with variables: open sentences (such as equations to be solved), generalizations (such as most theorems), and existence statements (such as most counterexamples, and some defining conditions, such as in the formal defining condition of "even" number: A number, n, is even if there exists an integer k such that $n = 2k$).

The distinction between open sentences and generalizations is reflected in the critical difference between equations and identities. Equations employing "x" may be about "x", but identities employing "x" are not. For example, "$3(x + 4) = 20$" is about "x", but "$3(x + 4) = 3x + 12$" is *not about x*. It is about the order relationship of addition and multiplication. Many times I have asked calculus and precalculus students, "What is this sentence *about*?" Most do not even understand the question. Very few know -- yet we expect students to read their texts when they do not even know what many of the sentences are about.

We can use language techniques to teach students to distinguish between these types of sentences and to recognize the importance of context. In English there are many definitions of the word "set" and readers use context to tell which is which. This feature of Mathematics is no different from English usage; it is the reader's responsibility to learn how to interpret sentences with similar appearance but different meanings. Now many students cannot interpret mathematical sentences, even in context, partially because they have not been taught to recognize the three basic categories. The current approach of devoting a paragraph in an algebra text to the fundamental subjects of open sentences and generalizations is insufficient and clearly not effective.

What *is* effective is to reorient a substantial fraction of homework and exam problems toward the goal of *understanding*, as opposed to just *doing*. Teachers know that students rapidly adjust their orientation toward any material to correspond to the requirements they perceive -- as defined by homework and exams. Every difficult language concept mentioned in this article can be and has been effectively taught to a variety of "math anxious" students by asking the students homework and exam questions which *emphasize* the concept (as opposed to numerical answers). Esty and Teppo (in press) present numerous examples in addition to those found throughout this article.

The essential difference between "free" and "bound" ("dummy") variables is likewise neglected. Variables quantified with "For all..." or "There exists..." are called "bound" or "dummy" variables. Dummy variables are placeholders which can be switched for other letters throughout a mathematical sentence without affecting the meaning (as long as certain conventional usages of notation are not violated). "For all x, $3(x + 4) = 3x + 12$" expresses the same meaning as "For all y, $3(y + 4) = 3y + 12$," since both variables are quantified. Both sentences apply to *all* real numbers. In English, the term "synonyms" refers to *words* with the same meaning; these are "sentence-synonyms" of Mathematics in which *sentences* have the same meaning. Other types of "sentence-synonyms" are discussed below.

On the other hand, the choice of letter *is* important in an open sentence. For example, "$6x = 48$" does not express the same meaning as "$6y = 48$," any more than "John is blond" expresses the same meaning as "Sam is blond." "x" is not quantified (it is "free") and cannot be switched.

Students study the content of sentences with variables daily in every mathematics course after beginning algebra. Surely they should study the forms and interpretations of these basic three types of sentences because they are essential to the language in which they will be working (Usiskin, 1988).

Here is a type of question not seen in high-school texts.

Exercise: Suppose each of the following sentences is true. Which express mathematical facts and which express facts which depend upon the particular things represented by the letters? a) $3(x + 5) = 12$; b) $3(x + 5) = 3x + 15$; c) $x(x + 1) = x^2 + 1$; d) ~~$S \cup T$~~ $S \subset T$; e) $S \subset S \cup T$.

Students need to be able to recognize that (b) and (e) are always true and therefore mathematical facts and that (a), (c), and (d) are not always true, and, therefore, if they are facts, they must be particular facts about whatever the letters represent. Note that the appearance of equation (c) is very close to that of an identity. An experienced reader may pause and wonder if "$x^2 + 1$" is a misprint for "$x^2 + x$." This proves the point that readers have expectations about what they read. But many algebra students do not know what to expect and have not even studied the relevant categories of sentences. Without training (or a lot of experience), students have difficulty detecting the difference between identities and other equations -- a difference which this example shows is often not indicted by the appearance of the sentence itself. Problems like these can be used to help students recognize the difference.

An essential preliminary language concept is the distinction between *equations* (sentences) and *expressions* (which play the role of words). Identities exhibit equivalent expressions, which are occasionally distinguished from other equations by using the symbol "$\equiv$" rather than "=". This type of equivalence is quite different from the equivalence of equations. The processes which apply to one do not necessarily apply to the other. For example, dividing through by four in the equation "$4x + 8 = 4x^2$" yields an equivalent equation, but dividing through by four in the expression "$4x + 8$" does not yield an equivalent expression. A problem which requests the complete factorization of the expression "$(2xy)^2 - 4x^3y$" often inspires techniques which are only applicable to sentences. Many students who know how to factor nevertheless obtain the wrong answer by omitting the "4" from the correct answer "$4x^2y(y - x)$," and many others also omit the "x^2" or "y", by canceling, a process which is not legitimate for expressions.

Logical Connectives. The Zero Product Rule stated in Example 1 employs the essential vocabulary "if and only if" and "or". This rule appears early in Algebra I, but the students are not prepared to grasp it. It could be restated, "$bc = 0$ is equivalent to $b = 0$ or $c = 0$," in which case students must then understand the term "equivalent" in place of "if and only if." Either way, important terms need to be explained.

In a language course the use of important words would provoke a discussion of their meanings. The same should be true in a mathematics class. There are only five key logical connectives: "and", "or", "not", "if..., then", and "if and only if". Proper understanding of the use of these words in mathematical sentences is critical, and misconceptions from imprecise English usages of these same terms must be addressed, or else the purpose of clear communication will not be served, because the

English and mathematical usages of these terms may differ. For example, students may erroneously solve "$|x| > 5$ with "$x < -5$ and $x > 5$," using "and" in an additive sense properly given to "or" in Mathematics. The mathematical use of "or" also needs some explaining, since it does not, by itself, imply exclusivity as the context sometimes does in English, for example, "I will move to New York or Los Angeles." "Or" and "and" can be virtual synonyms in English -- compare "Megan or John can help you," with "Megan and John can help you," which could have the same interpretation in English, whereas "or" and "and" definitely have distinct interpretations in Mathematics. The connective "not" is another word that is often mathematically misplaced in the context of English generalizations, as for example in, "All batteries are not alike."

The remaining connectives, "if..., then..." and "if and only if", are absolutely fundamental to mathematics, but not well-understood. In English we frequently use "if..., then..." with the meaning of the mathematical "if and only if." For example, "I will let you have the car tonight if you do your homework first," in which there is an implicit implication that if you do not do your homework first you will not get the car. Unfortunately, textbook authors frequently use "if..., then..." when "if and only if" would be preferable. A search of school and college textbooks reveals many examples of unnecessarily weak statements such as, "If $b < 0$ and $c > d$, then $bc < bd$." This is inferior to, "If $b < 0$, then $c > d$ iff $bc < bd$," because the former does not assert equivalence when it could. A valuable supplement to any text would be to note when theorems and definitions stated with "if..., then..." could (and should) be stated with "if and only if."

Generalization, Negation, and Existence Statements. The logic of negations, another language topic which is important in algebra. To show that a sentence is false, we note that its negation is true. Here is an example.

Example 3: Conjecture: "If $bc = bd$, then $c = d$." True or false?

Studying conjectures can help students learn to read mathematics critically and to distinguish truth from falsehood. This conjecture is an abbreviated generalization which omits the initial "For all b, c, and d...". Students who have studied generalizations know that this statement is an implicit generalization and the question asks if the sentence is *always* true. This understanding is triggered by the context in the same way that we recognize that "$(x + 1)^2 = x^2 + 2x + 1$" is an identity.

Because mathematicians regard a statement as false if and only if its negation is true; to show that this conjecture is false, we instead show that its negation is true. The thought process involves two distinct logical concepts. The first is the negation of a conditional sentence of the form "If H, then C." The second is the negation of a generalization, which is an existence statement -- the third basic type of mathematical sentence.

I have posed the conjecture "If bc=bd, then c=d" to many college students from freshman-level through seniors in mathematics education. A substantial majority (including the future teachers) do not regard it as false. Most would benefit from a study of the language concept of negation. If they were studying French, there would be a section on the use of negation. "Je sais." "Je *ne* sais *pas*." ("I know." "I do not know.")

The negation of "If H, then C" is "H and (not C)." The negation of the generalization "For all x, if H(x), then C(x)" is equivalent to "There exists an x such that H(x) and (not C(x))," that is, a "counterexample." The negation of a generalization is *not* a generalization. With a little training in the role of variables, students can understand that the negation of "For all b, c, and d, if bc = bd, then c = d" is "There exist b, c, and d such that bc = bd and c≠d." Then the example of the negation, b = 0, c = 1, and d = 2, serves as a counterexample to the conjecture, *proving* it false. When we offer this conjecture as a true-false question and mark a student's "true" response wrong, we frequently get an argument that it is sometimes (even "usually") true, since most students regard statements which are either true or false (but not both) as open sentence that are sometimes true and sometimes false. We can dispel the erroneous "sometimes" idea with the study of generalizations.

Examples 1 through 3 exhibit profound language concepts that affect the interpretation of *sentences*: representation, generalization, word order, logical connectives, abbreviation, and negation. It is not enough for courses to simply mention these special characteristics of Mathematics and then immediately assign traditional homework problems. These are fundamental characteristics of Mathematics which deserve class time, textbook emphasis, and homework of their own. The payoff is immense when students who truly understand the language are able to read, write, and think Mathematics correctly in years of subsequent courses.

Language Concepts in Equation-Solving

The process of solving an equation creates a mathematical *paragraph*. Two basic patterns from logic determine the organization of such paragraphs. Nevertheless, few students are exposed to the logic of solving equations.

Many students learn to manipulate an original equation to get new equations without understanding whether their steps preserve, add, or drop solutions. Consequently, many students are quite unsure of what they have learned about solving equations. There are only a limited number of theorems that need be applied to solve the usual sorts of algebraic equations (Esty, 1991, Chapter 4), and they are easy to express and easy for students to apply unerringly, but they must first understand generalizations and logical connectives. Examples 4 and 5 illustrate the two basic logical patterns of equation-solving.

Example 4: Solve (x + 5)(x - 2) = 2x(x - 2).

The imperative "solve" means "Find the values of x such that the sentence is *true*." The mathematically correct idea that such an equation can be regarded as an open sentence which is sometimes true and sometimes false startles many of my students. They think equations are true. They have not learned to distinguish between sentences which are mathematically true and open sentences like this one which are merely sentences expressed in mathematical notation. Thus the idea of "equivalence" cannot have full meaning for them.

Ideally the solution process consists of a sequence of equivalent equations, the last of which exhibits the solution(s). "If and only if" is the connective which connects equivalent sentences, that is, sentences which are true for the same values of the variable (This is a second type of "sentence-synonym" in Mathematics).

How many times do we see simple equations solved with the "=" sign misused as a connective? For example, The equation "5x = 60" may be solved with the written string, "5x = 60 = x = 12," in which the student *did* the right thing and *wrote* the wrong thing (Behr, Erlwanger, & Nichols, 1980). This misuse of "=" to mean, "I am about to do the next step," used to irritate me, but now I understand it better. At least, the student knew that the sentences "5x = 60" and "x = 12" *are connected*. If he or she uses "=" to connect them, it is only because we have not taught him or her how to properly express the connection. Study of the proper concept (equivalence of sentences) and a convenient symbol for it ("iff" or "⇔") is necessary to help solve that problem.

Continuing with Example 4, here is the "Theorem on Canceling": "bc = bd if and only if c = d or b = 0." It tells us that

(x + 5)(x - 2) = 2x(x - 2) iff x + 5 = 2x or x - 2 = 0.
iff x = 5 or x = 2 [by Uniqueness of Addition, twice].

The last pair of equations exhibits the solution.

Too many students will "cancel" the factor of x - 2 and accidentally drop a solution. There is no need for this to happen if the text states the Theorem on Canceling *explicitly* and the student's background includes enough familiarity with the language to permit full understanding of it. Correct solution processes often rely on understanding which forms are equivalent and which are not (Booth, 1989, p.57f). Clearly the concept of "equivalent" equations and sentences is fundamental to equation-solving, yet too many students manipulate equations without knowing what they are doing and why.

The next example makes clear the distinction between the connectives "if..., then..." and "if and only if."

Example 5: Solve $-(x-1)=\sqrt{x+11}$.

The approach is to square both sides to obtain the equation $x^2 - 2x + 1 = x + 11$. Next comes the equation, $x^2 - 3x - 10 = 0$, then $(x - 5)(x + 2) = 0$, and then, using the Zero Product Rule, $x - 5 = 0$ or $x + 2 = 0$. Then $x = 5$ or $x = -2$. Many students stop here.

Many students learn to check their solutions. Checking back (Why is this necessary? Didn't we do the steps right?) we find that $x = -2$ satisfies the original equation, but $x = 5$ does not. What happened? Why did it happen? This is a mystery which remains unsolved for too many students. They ask, "What did I do wrong?"

The answer is simple: The student did nothing wrong. The first step entailed squaring, which is an "if..., then..." process, not a process which guarantees equivalence. (The Rule on Squaring reads: If $f(x) = g(x)$, then $[f(x)]^2 = [g(x)]^2$.) What happened is concealed when algebra is taught without studying logical connectives and generalizations. Not all equation-solving steps yield the ideal case -- equivalence.

The Theorem About Extraneous Solutions tells us that, when an "if..., then..." rule applies, the solutions to the first equation are precisely the solutions to the second equation which *also* satisfy the first. Thus the solutions to the final equation in a sequence might have to be pruned down by checking them back in the original equation. School and college texts do not state such a theorem, nor do they relate the connective "if..., then..." to the phenomenon. Unfortunately, if the students have not studied "equivalent" equations and the role of the logical connectives, the checking back in the original equation that this theorem requires is easily confused with checking for *mistakes*, a process with which they are far more familiar. In some texts (for example, Ruud & Schell, 1990, Section 1.4) these distinct reasons for checking are juxtaposed without drawing a clear distinction between them. Thus, students who find an extraneous solution are naturally left with the feeling that they have done something wrong, but they do not know what. This is hardly the impression of mathematics that we wish them to acquire.

This feature of "if..., then..." also explains why a word problem may yield an equation which has a solution which does not actually solve the word problem. The point is, *if* the situation in the word problem holds, *then* the equation follows. This is not the same as equivalence, and there is no implication that *all* the solutions to the equation are relevant to the word problem. I know of no algebra text which makes this connection to basic logic.

Without proper emphasis on the connectives, students lose track of which types of steps are guaranteed to work and which are not. Our conception that mathematics is orderly and exact is belied by our examples when we avoid giving students the whole picture.

We should be careful when we state our rules. Suppose we are teaching students to solve simple equations and we state Uniqueness of Addition like this: "If $b = c$, then $b + d = c + d$." This looks good, is an axiom, and is often seen in algebra texts, but it is emphatically *not* the formulation which should be used to solve equations because it fails to assert equivalence when it could (by using "iff"). Here we expect students to infer equivalence when it is not explicitly stated, but the Rule on Squaring is stated with the same connective and yet there we fault them for errors which result from the same expectation. How are students supposed to distinguish subtle but significant differences in meaning if textbooks and instructors do not observe them? Precise thought requires precise language, and this version of Uniqueness of Addition is, unfortunately, only correct, not precise.

If we want our students to understand how to solve equations, we should do it right and emphasize the meaning of "equivalent" in the context of equations, give a convenient symbol for it ("iff" or "$\Leftrightarrow$"; the whole phrase "is equivalent to" is too long to employ regularly), supply the background in connectives they need, and then state our theorems so that equivalence is easy to see when it is there.

Again, the remedy to the problem is to change the emphasis of some of the homework. Sometime in every student's equation-solving career he or she needs some practice making connectives explicit and citing algebraic theorems to justify steps. This is a major part of geometry, but neglected in algebra. This is not the same as finding the solution, and not even the same as showing the work. It is to prove that students know *why* their work is appropriate and correct. After all, correct steps *are justifiable*. Citations help make connections between current homework problems and the theorems which express the essential similarities of many related examples. That is, citations encourage students to learn the language in which problem-solving patterns are efficiently expressed. Furthermore, they require students to reference and learn the corpus of known mathematics, which helps them avoid inventing their own convenient, but possibly incorrect, steps. Also, citations are good preparation for proofs, in which steps must be justified by reference to known results.

There are a limited number of common patterns in the application of logic to mathematics. We have just seen two in equation-solving. Understanding these patterns helps students better understand how to solve equations. Because equation-solving *is* a logical process, students should see homework problems in which equation-solving is treated not only as a skill, but also as simple logical process. To emphasize this I assign problems in which the instructions are to "exhibit every step, exhibit the connective, and cite a rule (theorem)."

I am convinced that the fruit of our neglect of logic is a pervasive feeling among students that equation-solving is mysterious. This is ironic, because we mathematicians feel that we have clearly and concisely stated the rules of algebraic manipulation. But tell that to students who have accidentally found extraneous solutions when they employed squaring. *They* don't think that mathematics is straightforward. The background that mathematicians have which makes mathematics understandable and mathematical equation-solving methods clear is denied to our students. Why? Because we do not teach them the language and, most essentially, the basic logic.

The Next Advance: Proof

Proofs are commonly regarded as more advanced than equation-solving. Our students have enough difficulty solving equations, so naturally we do not emphasize proofs in algebra. Should we mention proofs only in geometry? Or can students profit from proofs in algebra as well?

I agree that most students without any background in Mathematics are unlikely to benefit from proofs, because proofs are paragraphs in the language of mathematics, and students who cannot understand sentences will not be able to understand paragraphs. I also agree that there is no need to emphasize proofs the first time the students see algebra. But the usual curriculum gives the students two years of algebra. Don't proofs belong in there somewhere? How much more advanced are proofs than calculation-oriented problems?

The distinction between the skills needed to create a proof and the skills needed to solve equations is not so firm as is commonly supposed. Do students realize that they do a *proof* every time they properly employ steps which solve an equation?

In proofs the sentences are logically connected to form a paragraph. In equation-solving the sentences (equations) are logically connected (by "if and only if," as far as possible) to form a paragraph. The last sentence exhibits the solution(s). In each process, steps are justified by prior results (axioms, theorems, and definitions). In fact, a correct sequence of steps which solves an equation actually *proves* that the solution is correct. But we imagine these two processes to be quite different. Why? In equation-solving we have not taught them to see the logic they use. Rare is the text which has *any* problems which ask students to cite reasons and exhibit connectives when solving equations. But the fundamental idea of using previously-obtained results to justify steps is prominent in both equation-solving and traditional proofs.

The next example illustrates the additional knowledge and skills a student must have in order to follow, and even do, proofs in algebra. One step in many algebraic proofs is to restate the theorem with the logical connectives rearranged according to one of the basic logical equivalences. Then the proof proves the new, equivalent, theorem instead of the original theorem. This is very confusing for students who do not know the basic logical equivalences, especially since restatements rarely are mentioned explicitly.

Example 6: Consider the usual multiplication of real numbers. Suppose students already have axioms and results asserting that multiplication is associative, products are unique, $1b = b$ for all b, $b(0) = 0$ for all b, and that, if $b \neq 0$, then there exists a number, $1/b$, such that $(1/b)b = 1$. At this stage students can understand the proof of the Zero Product Rule if they know the basic logical equivalences (and they cannot if they do not). It is stated with "iff", so it can be regarded as having two halves by the logical equivalence of "A iff B" and "[If A, then B] and [if B, then A.]" This is one of the basic logical equivalences and students can easily learn that theorems stated with "iff" are often regarded as having two "if..., then..." halves. (Logical equivalences are a third type of "sentence-synonym.") One half of the Zero Product Rule is: If $bc = 0$, then $b = 0$ or $c = 0$.
Proof: If $b \neq 0$, then $1/b$ exists. Since $bc = 0$, $(1/b)(bc) = (1/b)0 = 0$. Also, $(1/b)(bc) = ((1/b)b)c = 1c = c$. Thus $c = 0$.

This proof does not begin with a hypothesis, and assumes something ($b \neq 0$) which is not a hypothesis. The logic is correct, but evident only to those who are logically inclined. Certainly many students do not have the background to fully understand this proof. However, if they knew a few basic logical equivalences they could recognize that this proof addresses a reorganization of the original statement.[1] Here is the theorem justifying that reorganization.

Theorem on "or" in the Conclusion: "If A, then (B or C)" is logically equivalent to "If (not B) and A, then C."

This is an abstract statement about a common pattern in which logical connectives are employed. Students can truly understand the proof only if they recognize the alternative, equivalent, pattern. The proofs of most of the rules for equation-solving employ logical equivalences to reorganize the original statement into the form in which they are actually proved. If every proof had a distinct pattern, then we might be justified in thinking that mathematical reasoning in algebra is too much to teach our students. But the number of basic patterns is only about a dozen and we should not pass up the opportunity to help students learn to reason mathematically.

The proof can be seen to begin with "not B" ($b \neq 0$ and use "A" ($bc = 0$) to deduce "C" ($c = 0$). Students can learn that results with "or" in the conclusion are usually reorganized using this logical equivalence. Otherwise, the organization of the proof does not make sense. The reasoning can be grasped in English by those students with good reasoning skills, but, by expressing it in Mathematics, it can be studied and learned even by those who do not begin with good reasoning skills.

For practice, students who have seen this theorem can be asked to employ it to deduce another fact from "If $x^2 > 25$, then $x > 5$ or $x < -5$." Or, after noting that fact, they can be asked what can be deduced about x given "$x^2 > 25$ and $x \not> 5$." We do not need to limit our students to reasoning in English when there is another language specifically designed to express the basic patterns of reasoning.

We spend a great deal of time categorizing algebraic equations. For example, students are trained to recognize "$3x^2 + 17 = x^2 + 2x + 9$" as a quadratic which is usually handled by consolidating all the terms on the left side. The quadratic pattern is important, but we spend more time on teaching students to recognize obscure algebraic patterns such as factorable cubics and quartics than we do on the few essential patterns of logic without which they are handicapped in mathematics understanding. To comprehend proofs in algebra students need to know transitivity of "if..., then..." (the most important tautology) and approximately a dozen of the most commonly used logical equivalences.[2]

The basic logical equivalences are fundamental patterns of mathematical thought which are easy to teach. Students can do the proofs with truth tables. Why not, in the process of studying the definitions and properties of the logical connectives, emphasize precisely those compound statements which appear regularly in mathematics, rather than assign homework problems consisting of arbitrary combinations of the connectives? Why not exhibit a simple proof and show how these combinations can be used to understand the proof? This approach may seem obvious, but it is not the approach used in logic texts or "finite" or "discrete" mathematics texts.

Translation. The language skills required to fully appreciate algebraic proofs have been mentioned above. Remarkably, only one more language skill, translation, is required to progress through the entire college (and even graduate-school) mathematics curriculum. Translation is mentioned here, not because the thrust of this article is high-level mathematics, but to complete the discussion of all the basic mathematical language skills. Translation skills can be taught whenever new terms are introduced.

Mathematics is a language and translation is frequently required. This article will not discuss the type of translation between English and Mathematics required to do word problems. That is an important, but separate, issue (Herscovics, 1988, p.63). The translation discussed here is *within* Mathematics. For example, one way to express a thought may emphasize symbols and technical vocabulary, while another may emphasize logical connectives. The ability to translate between these "dialects" of Mathematics is an important language skill.

Mathematics is notorious for its difficult terminology. Students often find that they don't really understand the terms, and, if they do, the proofs do not seem to use them the way they understand them. The reason is our attempt to do all the work in English, as far as possible. For example, students may attempt to understand a term such as set "intersection" by paraphrasing the given mathematical definition in English and memorizing that it is the set of elements "in both."

At some level it is good for the students to be able to translate into English; but it is equally important for them to have a firm grasp of the few essential mathematical connectives so that terms can be understood in their natural language -- Mathematics. A set is determined by its members, so the definition of "$S \cup T$" which determines its members, "$x \in S \cup T$ iff $x \in S$ and $x \in T$," is ultimately precise and the one utilized in mathematical paragraphs. Only by avoiding study of a few key logical terms do we force students to look to English to understand our vocabulary (Tall & Vinner, 1981).

Example 7: Suppose the students have just begun to study sets and they have learned these three definitions: 1) two sets are <u>equal</u> if they have the same elements, 2) S is a <u>subset</u> of T when all the elements of S are also in T, and 3) The <u>intersection</u> of S and T, $S \cap T$, is the set of elements in both. Suppose we state a few simple results, including

Result 1: If $S \cap T = S$, then $S \subset T$.

Some students can see that this must be true because intersecting with T does not remove any elements from S, so T must be at least as big as S. This sort of argument is correct, and may be convincing, but it is not what mathematicians call a proof. Furthermore, if we avoid proofs which use the full power of Mathematics at this elementary level, students will not receive the necessary practice with simple proofs prior to confronting complex results which truly require logic, translation, and reorganization.

Only a few weeks into freshman calculus students will be asked to understand "The limit of f(x) as x approaches c is L" which is equivalent to "For $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$." This sentence combines generalizations, an existence statement, and a conditional sentence. Our students should study these concepts individually first; this complex combination is not a good place to start. Nevertheless, many students *do* begin their exposure to proofs in analysis (as opposed to proofs in geometry) with this sentence. We should not be surprised at the blank responses we get when "ε-δ" proofs are introduced.

"If $S \cap T = S$, then $S \subset T$" can be proved as follows: If $x \in S$, then $x \in S \cap T$, by the hypothesis and the definition of set equality. Then $x \in T$ by the definition of intersection.

That is a fine proof, but students who learned only the given definitions cannot be expected to read and comprehend it because the steps in the proof do not have the same appearance as the information in the definitions. They have learned the set-theory terms in an important language, English, but not in Mathematics, the language which facilitates further advancement in mathematics. If those three definitions had been learned in Mathematics as well as English, the proof would be easy to grasp. It would require only translation of the terms and symbols in the theorem into the dialect of Mathematics which exhibits the use of the connectives, and reorganization of the statement into a logically equivalent form.

When first studying the concept "subset," a proof that, "Under hypothesis H, $S \subset T$" requires that the sentence "$S \subset T$" be translated into "if $x \in S$, then $x \in T$." Thus Result 1 has an implicit hypothesis in the conclusion. The reorganization typical to proofs of results with hypotheses in the conclusion is given by the basic logical equivalence:

Theorem on Hypotheses in the Conclusion: "If A, then (if B, then C)" is logically equivalent to "If B and A, then C.

In the conclusion of Result 1, "B" is "$x \in S$" and "C" is "$x \in T$. The rearranged result is, therefore, "If $x \in S$ and $S \cap T = S$, then $x \in T$. The proof addresses this version and begins with "If $x \in S$" as expected. It employs the definitions of intersection and set equality as stated in Mathematics (as opposed to English): $x \in S \cap T$ iff $x \in S$ and $x \in T$. $S = R$ iff {(if $x \in S$, then $x \in R$) and (if $x \in R$, then $x \in S$)}. Each step exhibits logical connectives, even though most of them were implicit, not explicit, in the original statement of the theorem.

Any version of a definition which avoids the logical formulation may be closer to English, but is further from Mathematics and further from the form in which mathematicians employ the concept. Therefore, students who learn that sets S and R are equal if their members are "the same" are likely to be uncomfortable with proofs in which the logic will derive from the mathematical formulation. The mathematical formulation appears tougher, but a little logic makes it clear and the logical version has the huge advantage that it displays the logical connectives the way they are likely to be used in subsequent mathematics. We are making regrettable sacrifices in future understanding every time we avoid Mathematics for an easy English alternative.

It is important to note also that logic applies to sentences, not expressions. Therefore our definitions should be regarded as definitions of sentences containing terms, not just definitions of terms. A definition of the *expression* "$S \cap T$" can be given in a version of Mathematics which emphasizes symbols and notation (in set-builder notation, $S \cap T = \{x | x \in S$ and $x \in T\}$), but it often must be reformulated as a definition of the *sentence* "$x \in S \cap T$" for use in a proof. If the students learn definitions in the version which exhibits the logical connectives (in addition to any English version that they find illuminating) they will have the tools to comprehend the *steps* in proofs. When we underline terms in definitions (supposedly identifying what we are defining), such as in the three English definitions in Example 7, we are misdirecting the students by emphasizing the term instead of the whole sentence.

Students of mathematics should learn to define *sentences*, not words. Sentences exhibit the contexts in which the terms are applicable. Even more importantly, when a student needs to understand a term, he or she will see it *in a sentence*. If the definition is expressed in a sentence-based version, an entire sentence containing an unfamiliar term can be replaced by an equivalent sentence which is expressed in more primitive terms. This translation makes the original sentence understandable.

The Fundamentals of Mathematics

What do students need to know that they do not now know which would allow them to truly grasp our terms, theorems, and proofs in algebra, set theory, and every other mathematical subject area? They first need to know about *truth* and *falsehood* and the five logical connectives. They need to know a dozen basic logical equivalences and a few tautologies. They need to know the three basic types of sentences with variables, especially the concept of *generalization* and its type of variable, a *placeholder*. Then, with this background, they need to know how to translate subject-area vocabulary into our language, Mathematics, so that the connectives which are so often implicit in mathematical results can be seen and manipulated. *This is not a long list.* Furthermore (and this is important), in addition to being essential to the full understanding of high-school algebra, it is sufficient for college and graduate-school mathematics as well. *This list contains all the basic language concepts of mathematics.*

Conscious reflection on what one is doing, and why, is an effective part of learning. Unfortunately, the mathematics curriculum has long been based on the theory that weaker students should receive a "cookbook" approach and that only the stronger students can learn the real "why" of math. I think this is false and misguided. I concede that, in the current curriculum, only the stronger students have even half a chance of figuring out the role of mathematical reasoning in algebra. This awful fact should be regarded as condemning our approach to teaching reasoning, not as condemning the "weaker" students. The "weaker" students may actually be potentially quite good, but unable to read their text, understand their teacher, or properly express themselves. I have taught a course based on the premises of this article for four years to many initially "math-anxious" college students (some of whom were honor students in other areas) who overcame their anxieties and past failures when they studied the language and the "why" of it. The results have been dramatic improvements in skills, concepts, and attitudes (Esty & Teppo, in press).

This article not only points out difficulties students have with numerous language aspects of Mathematics, it also gives a remedy for each. The treatment is to change some of the emphasis from *doing* to *understanding* by using effective homework and exam questions which emphasize understanding. Numerous additional problems can be found in Esty and Teppo (in press) and especially Esty (1991).

The reasons that Mathematics, the language, has not been taught are historical. As we all know, the curriculum in any subject area has its own inertia; the organization of subjects and the emphasis placed on them changes only gradually. Geometry was the first mathematical subject area to be logically organized (long before truth-table logic was developed) so it is the subject with which reasoning has been taught. Now that logic for mathematics is so accessible we might expect to approach the teaching of reasoning through the subject specifically designed to address the problem, but we do not.

Study of the language has been neglected for another simple reason. Mathematics is a foreign language. Fluent speakers of any language hardly notice the difficult constructions that trouble the beginning student. Word order, conventional usages, abbreviations, negations, and the like are second nature to college instructors; that is a consequence of years of practice. To teach these topics we must first pause and notice that they exist. Foreign language courses isolate such topics and devote sections to them. Unfortunately, mathematics textbooks do not devote much space to the language aspects of mathematics and, therefore, teachers are not expected to explain them.

These concepts *are* taught in "foundations of mathematics" courses, which have a very small audience of post-calculus pure mathematics majors. Thus, the study of the language has been omitted from the curriculum for most mathematics students -- even for those students who end up *teaching* math.

I have found that all the logic necessary and appropriate for mathematics and all the common patterns of mathematical expression are low-level and can be organized so that they are easily accessible to general students (Esty & Teppo, in press). Even math-anxious students can construct and extract information from truth tables. They can master sentences with variables. Furthermore, the students like the material because it helps them *understand* (not just *do*) mathematics. Many students -- good students in other subjects -- crave understanding, but don't get it in the usual curriculum (Witness the mathematics drop-out rate). Many students who took my course, "The Language of Mathematics," as a "last chance" to satisfy a university-wide mathematics requirement have commented that they "finally" understood math. "Why didn't they tell me this in high school?" is a common plaint.

It is easy to see why those who will continue in mathematics should learn the language, but what about general high-school students, including those who will drop out of mathematics? On the one hand, study of the language should help them assimilate algebra, so they will be less likely to drop out. On the other hand (and this thought applies to at least half of our students), for those who will not continue in mathematics, the exposure to patterns of mathematical expression and reasoning is more valuable than exposure to many minor and even some major algebraic methods. Everyone uses whatever reasoning skills he or she possesses every day; only a relatively small number of people employ the quadratic formula.

Mastery of the language in which mathematics is created is well-known to be essential for the *development* of mathematics, but its importance for the *expression* and *comprehension* of mathematical results has been underestimated. The language in which mathematical results are expressed, Mathematics, is worthy of study in its own right because it is fundamental to understanding the rest of mathematics and because it is accessible to secondary-school students who need the language to read, write, and think mathematics. It deserves to be integrated into the curriculum. This should occur as early as possible so that students can benefit from improved mathematics literacy from then on. An alternative would be an entire course devoted to the language of Mathematics. Whether or not students and the mathematics curriculum as a whole would benefit from such a course is an interesting question. I am personally convinced they would benefit greatly. To improve mathematics literacy something substantial must change.

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FOOTNOTES

[1]It is interesting to note that the proof does not need to address the case when "b = 0", as some algebra texts do. Clearly those texts are covering all bases for their students who do not know enough logic to know what is sufficient to constitute a proof.

[2]Explicit study of logical equivalences has benefits in unexpected places. For example, students who do not understand that the contrapositive of a statement is logically equivalent to the statement will have trouble with statistical reasoning, which is more subtle than mathematical reasoning. In mathematics, the truth of "If H, then C" tells us that its contrapositive, "If not C, then not H," is also true, so we can then assert that if C is false, so is H. In statistics, we are more likely to start with "If H, then C happens most of the time," and ask students to draw conclusions about H if C does not happen in the particular case being considered. In statistics courses many college sophomores do not understand the logic of this. Deductive mathematical logic is simpler and should be studied before inductive statistical logic.