Give a man a fish,
and you feed him for a day.
Teach a man to fish,
and you feed him for a lifetime.

– Chinese Proverb

CHAPTER 1

# Algebra is a Language

## Section 1.1. Reexamining Mathematics

This is a text for a language course. The subject is Mathematics, the language in which mathematics is written and thought. ("Mathematics" is spelled with a capital letter like "French" or "Japanese".) Like other languages, Mathematics has its own vocabulary, grammar (principles that govern the correct use of a language), syntax (the part of grammar that concerns rules of word order), synonyms, negations, conventions, abbreviations, and sentence structure. This course teaches them all.

Mathematics has some overlap with English, French, and Japanese, but Mathematics concerns topics that are beyond the capabilities of native languages. It is a specialized language with its own concepts and symbols that must be learned. Musical notation is another specialized language with its own concepts (such as *chords* and *sharps*) that must be learned. No language is self-explanatory. Even if you can carry a tune, you might not be able to read music.

Similarly, even if you can do some math, you might not be able to read math. Learning to read math takes work. (Learning to read music takes work. Learning to read English took you a great deal of work.) You already spent some time on Mathematics in algebra. Nevertheless

**Your experiences in previous math courses have not prepared you for the strong emphasis on reading and writing in this course.**

The homework is different because the goal is different.

**The goal is for you to become fluent in the symbolic language of mathematics so you can efficiently read, write, learn, and think mathematical thoughts.**

You should expect much less calculation and much more emphasis on how to think and express mathematical thoughts.

This text is more like a language text than it is like a typical math text. It teaches essential language concepts which have been neglected in the usual mathematics curriculum. It emphasizes the basic patterns of mathematical expression and thought.[1]

Language shapes thought. The language of mathematics helps us think mathematically in a way that English does not. Furthermore, a language need be learned only once and is then good forever after. Mastery of the symbolic language of mathematics will provide you access to the whole world of mathematics.

**Mathematics as a Language**. Is Mathematics actually a language–or is it really a body of knowledge, skills, methods, and theorems about numbers? Well, mathematics (the subject, with a lower case "m") certainly includes knowledge, skills, methods and theorems. But this course draws an important distinction between **what** is said and **how** it is said. What is said is mathematics (lower case). How it is said is Mathematics (the language, upper case). To see the difference, consider the technical, linguistic, definition of *language*.

**Definition 1.1.1**: Language: A non-instinctive system of communication using symbols possessing arbitrary (conventional, learned) meanings and shared by a community.

Most languages are spoken and the symbols are vocal–sounds. But for symbolic algebra the symbols are written. The language of algebra is certainly "non-instinctive"! It must be learned. It is a system of communication about mathematical objects such as numbers, sets, functions, operations, and equations. It is a language shared by a world-wide community of people who have an interest in the subject of mathematics.

Mathematics is a modern language. Most of our symbolic notation is less than 400 years old and even such basic symbols as "+" (plus) and "-" (minus) go back only 500 years. When algebra is done in a native language such as English or Greek, it is called "rhetorical" algebra. The subject of algebra is old. Some problems we now solve using algebraic notation were posed and solved by the Babylonians (before 2000 B.C., in the region now called Iraq), the Egyptians (before 1850 B.C.), and the Greeks (before 200 B.C.). However, their methods were very awkward and limited compared to modern methods.

Algebra did not progress very far until more modern times because the ideas of

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[1] Mathematical language skills are underemphasized at all levels through college. The National Council of Teachers of Mathematics has noted this problem, especially in grades 9-12, in its 1989 publication, *Curriculum and Evaluation Standards for School Mathematics*.

algebra are hardly expressible in ordinary language. The symbolic language of algebra is critical for algebraic thought. Ancient peoples certainly were smart enough to have mastered algebraic thought, but algebraic language hadn't been invented, so they had very few algebraic ideas. Being “smart” is not the point.

When the language became available, there was an explosion of new mathematical results. The language was an effective tool for thinking. Sometimes even simple tools can make a big difference. For example, the stirrup is simple and very effective in helping keep a rider on a horse. It is hard to imagine why the Greeks and Romans rode horses for over 1000 years without inventing it. Amazing! (It appears in Western civilization sometime after 400 A.D.) The idea of the language of algebra eluded mankind for even longer. Algebraic language now appears simple to those who understand it, but it is clearly profound and difficult for those who have not studied it.

It takes some effort to learn to ride a horse, and it takes a lot of effort to learn to read English. It takes even more effort to learn to read English well enough to learn new things by reading English. In this course you will learn to read Mathematics well. You will **be able to learn math by reading math**. Of course, this progress will take a substantial effort on your part.

Linguists (people who study how language works) agree that it is difficult to have thoughts without the proper language in which to express them. Language behaves like a groove, or a prepared track, down which your thoughts will tend to go.

The implications of this “groove” theory of language are vast. For Mathematics, it tells us you cannot expect to think algebraic thoughts until you learn the language, regardless of how “good at math” you might or might not be. If you have not been taught algebraic symbolism, you cannot know how “good at math” you really are!

There are numerous stories of famous mathematicians who hated arithmetic and who got low grades in mathematics in grade school. They blossomed only after they learned the language. Unfortunately, our school system has not yet made the language of algebra a high priority and this course is likely to be your first exposure to the study of the language (as opposed to the results expressed in that language).

**Reexamining Mathematics**. As we progress in mathematics we constantly reexamine things we learned and regard them in new ways.

Example 1: You began to reason abstractly in mathematics as a child. As a toddler you learned to say the sounds “1, 2, 3, 4, 5” in order. The smiles of your parents told you that you had done something right. At first they were just sounds you had memorized; you were not really counting objects. With a lot of guidance and effort, you learned to count. Then the number *five* became a way to describe a group of objects, for example, “five blocks.” Your concept of *five* changed. Instead of just a pleasing sound, it became an adjective.

Still later your concept of *five* changed again. A group of five blocks and a group of five marbles have something in common. Children who are just learning to add and who count 5 blocks and 2 blocks to get 7 blocks have to start over again when asked about 5 marbles plus 2 marbles–after all, the objects are not the same!

You learned you can add "5 + 2" without knowing whether the objects are blocks or marbles. Your concept of *five* became a noun–a mental object in its own right. It was abstracted from many instances where *five* was an adjective. You can touch five blocks, but you can not touch *five*. It is a new reality at a higher conceptual level.

*Five* retains its important role as an adjective, and, in addition, serves as a noun (a mental object). New concepts allow flexibility. This conceptual advance is essential to mathematics.

**Definition 1.1.2**:[1] A notion or idea is abstract when it summarizes the essence of many similar examples by regarding the similarity as one new mental object. The abstract idea is called a concept.

You can see a pile of five blocks. You cannot see *five*. Numbers are abstract.

Example 2: When you were young you learned to subtract. At first you thought of it as "take away." Three take away one is two. 3 - 1 = 2. However, you cannot take 5 rocks from a pile of 3 rocks, so "3 - 5" is not possible. This was regarded as completely obvious for thousands of years.

Of course, now we know you can subtract 5 from 3 using the concept of negative numbers. Negative numbers are a big part of daily life because we all use the concepts of credit and debt.

The ancient Greeks did a lot of excellent mathematics, but since then we have had to reexamine what numbers are. The Greeks lacked number lines which could be used to illustrate 0 and negative numbers.

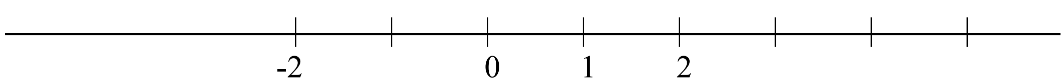


**Figure 1.** A number line, with positive numbers to the right of 0 and negative numbers to the left of 0.

[1] Important definitions, theorems, and comments are numbered in **bold** print in a single sequence. Learn everything in bold. Examples are numbered in their own underlined sequence.

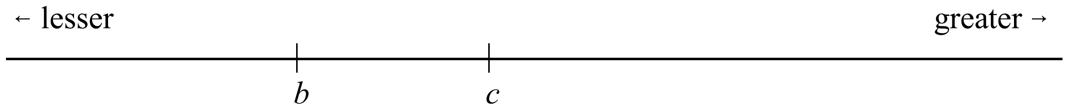


**Figure 2**: $b < c$. This is read "$b$ is less than $c$."
On the number line it means "$b$ is to the left of $c$."
It does not necessarily mean that $b$ is "smaller" than $c$ because
$b$ could be large and negative.

Now children know about 0 and negative numbers–things that even the smartest Greeks did not know. Being smart is not the point. You just have to learn new tools, such as the number line and algebraic notation, to help you think about new mathematical concepts. (We will use number lines extensively in later sections, especially in Section 1.5 on numbers, Section 2.1 on sets, and Chapter 3 on logic.)

**Algebra**. As a young child you revised your thoughts about what *five* is (Example 1). Later you revised your thoughts about what numbers are (to include negative numbers, Example 2). When you take algebra you need to revise your thoughts about what mathematics really is.

In arithmetic, mathematics is largely the study of numbers. The sentences of arithmetic are about numbers: $3 + 2 = 5$. $4 \times 5 = 20$. However, in algebra many of the sentences are not about numbers. The focus shifts to operations and order.

**Algebra extends arithmetic by also emphasizing operations and order.**[1]

In algebra we study operations such as addition and multiplication and the order in which they are done. Operations and order are mathematical concepts that are more sophisticated than numbers.

These concepts are abstract and hard to develop unless you have the proper language, the symbolic language of algebra. We spend all of this course developing the concepts and the necessary language simultaneously.

**The Components of the Language**. Here we define some of the basic terms of the language. Like other languages, Mathematics has nouns, pronouns, adjectives, verbs, and sentences.

**Definition 1.1.3**: Expressions are the nouns and pronouns of Mathematics.
**Definition 1.1.4**: Variables are letters used to represent, or hold the position of, mathematical objects.

[1] Major points are in **bold**. Strive to understand everything in bold.

Example 3: Expressions include all numbers, such as "7" and "$5(3^2)$", which are nouns.

Expressions such as "$x$", "$ab$", and "$x^2 + 2x + 1$" have variables.

"$x = 2$" is not an expression–it is an equation. It is a sentence.

**Definition 1.1.5**: A sentence is declarative when it makes a statement.

**Definition 1.1.6**: An equation is a declarative sentence in which the verb is "=" [pronounced "equals".]

Sentences which are true or false are declarative.

Example 4: "Chicago is in Illinois" is a declarative sentence. It is true.

"Chicago" is an expression (a noun), but not a sentence. It, by itself, is not true or false.

"$3 + 4 = 7$" is a declarative sentence. It is true.

"$3 + 4$" is an expression, but not a sentence. It is not true or false.

"$3 + 4 = 10$" is a declarative sentence. It is false.

The truth of a declarative sentence with a variable may depend upon the value of the variable.

Example 5: "$x + 2 = 5$" is a declarative sentence. It is true when "$x$" represents 3 and it is false otherwise.

"$x + 2$" is an expression and not a sentence. It is not true or false.

"$2x + 3x = 5x$" is a declarative sentence. It is always true, regardless of the value represented by $x$.

"$x = x + 2$" is a declarative sentence. It is always false.

**A Preview of the Uses of Variables**. Think of the rest of this section as a preview. All of the terms and concepts introduced in these introductory examples will be discussed very thoroughly later.

We must distinguish two distinctly different usages of variables in Mathematics. Unfortunately, two sentences may appear similar but have radically different meanings. Two similar sentences may even be about radically different types of mathematical objects.

Example 6: Consider the sentences:

(1) "$2x + 3x = 15$"

(2) "For all $x$, $2x + 3x = 5x$" and

(3) "$2x + 3x = 5x$." [This is an abbreviated version of sentence (2).]

Sentence (1), "$2x + 3x = 15$," gives information about the value of $x$; Sentences (2) and (3), "$2x + 3x = 5x$," do not. They are similar in appearance, but very different in meaning.

You can solve Sentence (1), "$2x + 3x = 15$, " and determine the only value of $x$ that makes it true ($x = 3$).

**Definition 1.1.7**: To solve an equation is to find the value(s) of the unknown that make it **true**.

**Definition 1.1.8**: A variable is an unknown when it represents a number or numbers which you are supposed to determine.

In the equation "$2x + 3x = 15$" the letter $x$ is an unknown. In its solution, "$x = 3$," the value of $x$ is no longer unknown, but we still call it an unknown anyway.

Sentence (2), "For all $x$, $2x + 3x = 5x$," is a true generalization because the component sentence (3), "$2x + 3x = 5x$" is true for all values of $x$.

**Definition 1.1.9**: A generalization is a declarative sentence that asserts that something is always true. When a mathematical sentence has a variable, by "always" we mean "regardless of the value of the variable." [We will discuss generalizations thoroughly in Section 4.1. Generalizations are not necessarily true.]

In contrast to the equation "$2x + 3x = 15$," the equation "$2x + 3x = 5x$" is always true. If you were asked to solve it, you would find that every value of $x$ works. No matter what expression is substituted for $x$ in "$2x + 3x = 5x$," the sentence remains true:

$2b + 3b = 5b$.
$2x^2 + 3x^2 = 5x^2$.
$2(y - 4) + 3(y - 4) = 5(y - 4)$.
2 oranges plus 3 oranges is 5 oranges.
$2\,f(x) + 3\,f(x) = 5\,f(x)$.

You do not have to know anything about "$f(x)$" to know the last line is true. It is the pattern that matters, not the symbols placed in that pattern. In the identity "$2x + 3x = 5x$," the letter $x$ is used to hold places in the pattern of operations. It is a placeholder, also known as a dummy variable.

**Definition 1.1.10**: An equation with a variable (one or more letters) is an identity when it is true for all values of the variable. An equation with a variable is said to be a conditional equation (and an open sentence) when its truth depends upon the value of the variable.

More on Example 6: The equation "$2x + 3x = 5x$" is an identity. It is always true.

The equation "$2x + 3x = 15$" is a conditional equation and an open sentence. Whether is it true or not depends upon whether $x = 3$ or not.

**Convention 1.1.11**: If a mathematical sentence with a variable is always true, then it will be regarded as a generalization even if it is not explicitly asserted that it is always true. Therefore, identities (and other mathematical sentences that are always true) will be regarded as generalizations.

Sentence (2), "For all $x$, $2x + 3x = 5x$," is explicitly a generalization. Sentence (3), "$2x + 3x = 5x$," will be regarded as a generalization, even though it is not explicit that it is one.

**Definition 1.1.12**: Variables in generalizations are placeholders. Placeholders are also known as dummy variables. A characteristic of a placeholder is that it may be replaced by any expression of its kind (say, any number). Placeholders are used in sentences about operations and order (as opposed to sentences about numbers). [We discuss placeholders again and again.]

**In algebra, Mathematics uses expressions with variables to represent both *numbers* and *sequences of operations*.**

Example 6, again. The sentences "$2a + 3a = 5a$, for all $a$" and "$2x + 3x = 5x$, for all $x$" look different, but they say **exactly** the same thing. The role of "$a$" is to **hold the place** of any number or expression in a sentence about how the operations "Multiply by 2" and "Multiply by 3" and addition relate to the operation "Multiply by 5." That is why we call the variable a placeholder.

When you have **one** abstract expression of the pattern, you have it all. As soon as you grasp the meaning of "$2x + 3x = 5x$, for all $x$" you know all the other versions that use "$a$" or "$b$" or oranges. There is nothing more to say. The pattern says it all.

The language of mathematics, which is essentially the same as the language of algebra, is well-designed to discuss patterns of mathematical operations.

**Math literacy depends largely upon pattern recognition.**

Distinguish these two uses of variables:

**Unknowns represent numbers in sentences about numbers.**
**Placeholders hold places in sentences about operations and order.**

Example 7: Consider the sentence, "For all $x$, $x^2 \geq x$."

It is a generalization–a false generalization.

Do you see why it is false?

There are no exceptions among the integers, but $x = 1/2$ is an exception. When $x = 1/2$, $(1/2)^2 = 1/4 < 1/2$. So, in this case, "$x^2 \geq x$" is false. This is algebra, and "$x$" is a placeholder which holds the place of any real number, and real numbers include fractions. The generalization is false.

It may be that "$x^2 \geq x$" strikes you as "usually" or "almost always" true, but that does not matter in a generalization. Either the component sentence "$x^2 \geq x$" is **always** true, or not. This one is not. The generalization is false.

In this case the example $x = 1/2$ is called a "counterexample" to the generalization.

**Definition 1.1.13**: An example that proves that a generalization is not true is called a <u>counterexample</u>.

<u>Example 8</u>: True or false (as a generalization)? "$a - b = -(b - a)$."

This is an identity. It is always true, regardless of the values of $a$ and $b$. As a generalization, it is true.

<u>Example 9</u>: True or false (as a generalization)? "$2x \geq x$."

This is false! When $x = -1$, it says "$-2 \geq -1$" which is false. "$x = -1$" is a counterexample.

<u>Example 10</u>: Some equations are always true, some are not. The equation "$2(x + 1) = 10$" is not always true. It determines a value for $x$. If $x = 4$, it is true, otherwise it is false. The variable $x$ is an unknown–a particular number.

In contrast, the equation "$2(x + 1) = 2x + 2$" is **always** true. It does not determine a value for $x$. That is because the equation is not **about** $x$, a number. On the contrary, it is about a completely different type of mathematical object It is about the operations of addition and multiplication. Think of this as an abbreviation of "For all $x$, $2(x + 1) = 2x + 2$" which is a symbolic way of saying "Add one and then multiply the result by two" yields the same result as "Multiply by two and then add two." The English version uses no numbers. The symbolic version uses a placeholder.

**<u>Abstract Mathematical Objects</u>**. Mathematics is a language designed to discuss mathematical objects, and the objects of algebra are abstract and best discussed with symbols, not English. Here are two types of abstract mathematical objects.

Example 10, "$2(x + 1) = 2x + 2$," treats "Add 1" as a thing–a concept in the category of functions. We use an entire section (Section 2.2 on Functions) to help you understand how a command such as "Add 1!" can be a mathematical "thing".

Mathematical methods are also "things"– mathematical objects.

**Definition 1.1.14**: A <u>method</u> is a set of instructions for doing some type of problem.

Example 11: The formula for the circumference of a circle is "$C = \pi d$" (pronounced "$C$ equals pi $d$") where $C$ stands for the circumference and $d$ the diameter. The information in the sentence "$C = \pi d$" is about a method, not a number. It answers the question, "How do you find the circumference of a circle?"

The circumference depends upon the diameter, but the **method** does not.

There is **one** method that solves **many** problems. The method is an abstraction. The language of algebra allows us to write methods. This one is simple, "$C = \pi d$."

In English you might say, "To find the circumference of a circle take the diameter and multiply by pi."

Usually instructions are in the imperative, "Do this!"

**Definition 1.1.15**: A sentence is imperative when it expresses a command.

"Take the diameter and multiply by pi," is an imperative sentence. "$C = \pi d$" says the same thing in symbols. In Mathematics, instructions are written in the declarative (as statements of fact). The method has a problem-pattern and a solution-pattern.

$$\text{problem-pattern} \rightarrow C = \pi d \leftarrow \text{solution-pattern}$$

The left side gives the problem-pattern, "What is the circumference?" and the right side gives the solution-pattern, "Multiply the diameter by $\pi$." The method (imperative) is stated as a fact (declarative).

**Identities State Methods.** Formulas are one way to express methods as facts. Identities are a second way. Here is an example that describes how to do a process from arithmetic.

Example 12: Find 5 - (-2). [Pronounced, "Five minus negative two.]

Solution: $5 - (-2) = 5 + 2 = 7$.

[Pronounced, "Five minus negative two equals five plus two equals seven."]

An identity expresses the method. The problem requires subtracting a negative number. How do you subtract a negative number from a positive number?

You can express the method in two ways:

a) Express the method in English (as a command).

b) Express the method in mathematical notation (as a statement of fact).

Answer: a) "Drop the negative sign and add."

[This thought can be expressed with different words.]

b) "$b - (-c) = b + c$."

[This thought can be expressed with different letters.]

Pronunciation. You must learn how to read Mathematics aloud. After all, it is a language.

"$b$ - (-$c$) = $b$ + $c$" is read "$b$ minus negative $c$ equals $b$ plus $c$."

You might also hear, "$b$ minus minus $c$ equals $b$ plus $c$." I say it both ways, but the first way using the word *negative* is better. *Minus* is used to express an operation, subtraction. *Negative* is used to express a sign change. They are not the same. Calculators have different keys for the two different operations. Nevertheless, the words *minus* and *negative* are so closely related that many mathematicians say *minus* when they really mean *negative*.

**Problem-Patterns and Solution-Patterns**: The sentence "$b$ - (-$c$) = $b$ + $c$" is an identity. It is a special type of equation and also a theorem (because it can be proven). It may not look like a command, but there is a plus sign on the right side and that does tell you to add. The **left** side of the identity expresses a problem-pattern and the **right** side tells you the solution-pattern, which tells you what to **do**.

Example 12, continued: Many problems have the same abstract pattern:

$$5 - (-2) = 5 + 2$$
$$16.1 - (-32) = 16.1 + 32$$

problem-pattern → $b - (-c) = b + c$ ← solution-pattern

The letters do **not** represent particular numbers. (The identity does not specify $b$ or $c$). The letters merely **hold places** in patterns of operations where any number could go.

**Definition 1.1.16**: Methods can be expressed in declarative sentences with problem-patterns and solution-patterns. The problem-pattern is an abstract description of the type of problem using a letter or letters, and the solution-pattern gives the corresponding pattern of operations used to do the problem. (Usually, but not always, the problem-pattern is on the left. In an identity, the two sides are equal and may be switched.)

**Learning a Language**. If you were learning German, you would not expect to be very good after one week. However, the most common vocabulary words would appear frequently and the basic sentence structures would reoccur every day. You would get better and better at German.

Keep this in mind as you learn Mathematics in this course. It really is a language course! The most common vocabulary words appear frequently and the basic sentence structures reoccur every day. You will have many chances to understand everything because you will see all the topics many times. We never drop a topic.

Chapters 1 and 2 introduce mathematical concepts and the language simultaneously, with a great deal of English explanation. During Chapters 3 (Logic) and 4 (Sentences) it will all come together. If something is confusing in an early chapter, do not worry. We will visit it again. You will get better and better.

**Conclusion**. This is a text for a language course. You should expect homework that is different because it emphasizes language topics.

The language of algebra is not instinctive. Like other languages, it must be learned.

Algebra is not only about numbers, but also about operations and order.

Methods of mathematics can be stated as commands in English, "Do this!" However, Mathematics states methods as facts–declarative sentences with a problem-pattern and a solution-pattern, using variables as placeholders to express the patterns.

Throughout this text you will be held responsible for terminology. Learn the definitions of the terms we discuss. (The key terms are listed at the end of the conclusion of each section.) You may already have a vague mental image of what these terms mean, but vague thoughts are not good enough. **Learn precisely what our terms mean so that you know precisely what we are talking about. Our terms are fundamental concepts of the language.**

Do not be too impatient. Fluency in any language takes time and practice. In this course, fluency is not expected until near the end of Chapter 4 (many weeks away!). This course is organized and paced so that if you read the text and do the homework, you will learn to comprehend and write Mathematics–regardless of whether you imagine yourself to be "good at math."

Terms: Abstract, concept, expression, variable, declarative, equation, solve, unknown, identity. Other terms that we will discuss far more thoroughly later include: generalization, placeholder, counterexample, method, imperative, problem-pattern, solution-pattern.

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## Exercise for Section 1.1 "Learning the Language":

**Note** on * and ☺: * means the answer is important and should be memorized.
☺ means the problem is very short and could be done aloud in class.

A1. In previous math courses did you (personally) learn primarily by *reading* your text or by *doing* what your teacher showed you how to do? [Or some other way?]

A2.* This text distinguishes "Mathematics" from "mathematics." What is this fine distinction?

A3.* True or False: Methods can be expressed as facts.

A4. What makes something *abstract*?

^^^^ ☺ Grammar. Categorize these as either expressions or sentences.

| | | | | |
|---|---|---|---|---|
| A5. | a) $3x + 5$ | b) $x = 2x - 7$ | c) $x^2$ | d) $x = 5$ |
| A6. | a) $x + 5 = 12$ | b) $x + 2x = 3x$ | c) $x + 2x$ | d) $x = 14$ |
| A7. | a) $2 < 6$ | b) $2 + 6$ | c) $2 + 3 = 5$ | d) $2x + 3$ |
| A8. | a) $2 + 4$ | b) $x < 4$ | c) $x^2 < 9$ | d) $x + a$ |

^^^^ ☺ Give the formula for

A9. The circumference of a circle.

A10. The area of a circle.

A11. The area of a square.

A12. The perimeter of a square.

A13. ☺ True or false? Generalizations are necessarily true.

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B1. Comment on whether you should acquire mastery of the material in each section of this text before beginning the next section.

B2.* a) Thinking of Mathematics as a language, what parts of speech are expressions?
b) What part of speech is "="?

B3.* ☺ What is the term for an equation that is always true, regardless of the value of its variables?

B4.* ☺ What is the term for a sentence with a variable that asserts that something is always true?

B5.* ☺ What is the term for an example that proves a generalization is false?

B6.* ☺ Suppose an equation is given. What is the term for the variable for which you are supposed to solve?

B7.* ☺ What makes a sentence declarative?
B8.* ☺ What makes a sentence imperative?

B9.* Students who think that algebra is only about numbers must expand their thoughts to understand that algebra is also about ____________________.

B10. a) In the discussion of abstract numbers in Example 1, the number *five* treated as an adjective was abstracted into a number treated as a _____________.
b) In the discussion of methods and identities (Example 1.1.12), methods treated as commands are abstracted to methods treated as _____________________.

^^^^ Here are theorems (identities are a type of theorem). For each, create one example that fits the problem-pattern and use the identity to find the value.

B11. $c - (-a) = c + a$

B12. $a - b = -(b - a)$

B13. $(a/b)(c/d) = (ac)/(bd)$

B14. $-a - b = -(a + b)$

B15. $a/(b/c) = ac/b$

B16. $(a/b)/c = a/(bc)$

^^^^^^^^

C1. Read the definition of *language* and say why Mathematics is a language.