Instructor's Manual

for

The Language of Mathematics

Nineteenth Edition

by Warren W. Esty

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Why Read This Manual?

This manual is intended to teach you, the instructor, some of what you would know about the course if you had already taught the course a couple of times. You have never taught another math course like this one, and your students have never taken another math course anything like this one. Take my advice and read some of this manual.

Note the

- goals (which are quite different from the goals in most math courses and therefore must be clearly articulated to the students. See the Preface, page iii in the text and page 4 of this manual)
- language approach (again, you and the students must adjust. See the preliminary section "To the Student" on page v in the text)
- effective way to deal with students who have "math anxiety" (See page 38 of this manual)
- good problems to ask in class (conveniently marked in the text with ☺)
- **section-by-section comments** for the instructor, including:
  things to emphasize
  what students find most difficult to grasp
  how students react to the material
  excellent problems to ask in class after lecturing

Many problems are designed to be asked in class and answered aloud immediately. They will help students with:

- pronunciation (e.g. 1.2, A25-72; 1.3, A9-24; 2.1 A5-14, 2.2, A33-38)
- grammar and syntax (e.g. 1.3, A25-34; 1.4, A21-34; 2.1, A35-64)
- interpretation of symbols (e.g. 2.2, A4-7, A29-32, B9-12)

I strongly recommend you use such questions (marked in the text with ☺) in class to help your students learn to read Mathematics. Short, meaningful symbolic sentences can be read in class very rapidly. Students can then be asked about their meaning and everyone learns from their responses and your feedback. This is a reading course and it is appropriate for students to learn to read and respond to mathematical sentences rapidly.

The text emphasizes simple things that other texts take for granted, such as correct pronunciation of symbolic sentences. Students will not become comfortable with Mathematics if they cannot even read the sentences! Going over problems aloud in class helps the class learn these essential basics, and helps you, the instructor, learn how uncomfortable or comfortable they are.

General Observations

This is a language course. Students have a lot to learn about the language including

- pronunciation
- order of operations
- operations as objects of thought and subjects of sentences

  Many don't even know the subject of sentences such as "$3(x + 4) = 3x + 12$." Students who think that sentence is somehow about "x" have a lot to learn in Chapter 1.
- placeholders (dummy variables)
- methods as objects of thought and subjects of sentences
- how methods can be expressed using patterns with variables
- grammar
- the correspondence between different letters such as "x", "f", and "S" and the mathematical objects they represent
- connectives (for example "if ..., then...".)
- truth
- generalizations
- existence statements
- relations (=, <, the connectives "$\Rightarrow$" and "iff", and functions)
- terms (for example, *subset* and *union*) can be defined with different approaches: in English, mathematical symbolism, or using logical connectives
- concept definitions (as opposed to vague concept images)
- replacing sentences with other, more elementary, sentences

  For example, you may replace "$|x| < c$" with "$-c < x < c$." You may replace "$S \subset T$" with "If $x \in S$, then $x \in T$."
- how to do mathematics in the contexts of numbers, sets, functions, and equations
- how to use variables to set up word problems
- justification

  When solving an equation, successive equations are connected and the steps are justified by general results which can be expressed abstractly.
- patterns of mathematical reasoning
- deduction and proof

The Chapters

Chapter 1 introduces many of the features of the language of mathematics. The ideas, terms (vocabulary), and theorems (sentences) illustrate basic language concepts, all of which continue to be analyzed throughout the text.

Language requires vocabulary. Chapters 1 and 2 are like the opening chapters of a beginning French (or Japanese, with it's strange symbols) course in which dialogues properly utilize French words in grammatical sentences. Students must study the dialogues and translate the vocabulary words. They begin, but only begin, to absorb the proper sentence structure. How would a French text proceed if no sentences could be written until after all the vocabulary had been mastered?

Chapter 2 introduces more abstract concepts such as sets, functions, equation solving (which use connectives and operations on equations), and word problems (which use functional thought).

Chapter 3 uses truth tables to analyze connectives and how they relate to the most common patterns of mathematical sentences and reasoning. Examples from Chapters 1 and 2 help illustrate the logical patterns we want students to grasp.

Chapter 4 finishes the discussion of variables, including the use of quantifiers. By combining the subjects of the first two chapters, the logic of Chapter 3, and the quantifiers in Chapter 4, students should be able to comprehend any mathematical sentence, including theorems and definitions defining new terms or using new symbols. See Section 4.6 for a summary covering to the end of Chapter 4.

Chapter 5 is on proofs.

Pronunciation. Have your students pronounce mathematical expressions out loud in class. They will have trouble reading and understanding them if they can't even pronounce them. Appropriate problems are marked in the exercises (with ☺).

Conjectures. One of my favorite teaching tools is the conjecture. I put a mathematical sentence on the board explicitly labeled "conjecture" (e.g. "Conjecture: $x < 5 \Rightarrow x^2 < 25$") and ask if it is "True or false?" (This one is false, because x might be negative and less than or equal to -5, say, -10.) Usually the conjecture resembles something true that they have seen. Many of my conjectures are false. Conjectures help students learn to read with precision and to learn that not everything is true. Conjectures are a great teaching tool to teach critical thinking–to transfer responsibility for truth from the authority (you) to the student.

Section-by-Section Comments

Chapter 1. Algebra is a Language.

This is a language course. In Chapter 1, students learn vocabulary, pronunciation, parts of speech, and how sentences are constructed. Often the subjects of the sentences are mathematical operations, not numbers. Operations are mathematical objects that are more abstract than numbers. Students already know many properties of numbers; now they learn properties of operations and learn to use the language to conceptualize operations and express their properties.

In Mathematics, the same symbols are used in two different ways to express thoughts about

- numbers (and, in Chapter 2, sets, functions, and equations)
- operations on, and relations between, mathematical objects such as numbers, sets, functions, and equations

Chapter 1 distinguishes these two different levels of abstraction. Students learn the use of mathematical patterns and placeholders to express thoughts about operations. They also learn that the emphasis of the course is really on the language, not on obtaining numerical answers.

Conventions. This text distinguishes between the uses of lower- and upper-case letters. Variables for numbers usually are lower-case letters such as x, y, a, b, and c. (Some well-known formulas such as "$C = \pi d$" use upper-case letters for numbers.) The letters n, i, and j are reserved for integers. Sets are represented by S and T and upper-case letters near them in the alphabet. Sentences in logic are represented by upper-case letters such as H, C, A and B. Making the distinction firm helps students read mathematical sentences.

Section 1.1. Reexamining Mathematics.

This section defines *language*. It describes parts of speech of Mathematics, the uses of variables, and begins to describe how procedures of arithmetic can be expressed as algebraic facts using placeholders.

This section begins to open the student's eyes to the ways we express mathematical methods. It makes the connection between *doing* math and expressing *how to do it*. Methods are general and can be expressed symbolically. We write mathematical methods as theorems which are facts. Students need to know this to be able to read math. They like this section, but, of course, they cannot fully grasp it the first day. Some students may even need Section 2.3 on equation-solving for them to fully understand the connection between *doing* and *facts*.

Lesson: Mention the terms listed at the end of the conclusion to the section. Each section has a conclusion and each conclusion is followed by a list of terms worth knowing. For example, in this section, the list includes: abstract, concept, variable, declarative, equation, solve, unknown, and identity. These are terms they should learn right away. The list continues with terms that will be discussed more later. For example, *problem-pattern* and *solution-pattern* are useful for discussing how we write about methods of doing problems.

Cover two major ways to express methods:

1) Formulas,

2) Identities (equivalence of expressions),

[We will discuss a third way, theorems that relate equations, in Section 1.4. Here is an example of the third way: $x + a = b$ iff $x = b - a$.]

Do this by abstracting the patterns after displaying several examples.

e.g. 1) $C = \pi d$ or $A = \pi r^2$. 2) $1a = a$. $a + 0 = a$. $(a/b)(c/d) = (ac)/(bd)$, $(a/b)/(c/d) = (ad)/(bc)$. (Section 1.4 has many examples)

Identities and theorems that relate equations give their own problem-patterns and solution-patterns. Formulas are abbreviated results and require knowledge of the context which is not stated explicitly.

Expectations. I recommend you tell them you do **not** expect them to understand everything as they go along (unlike other math classes). Many students are not sequential learners the way almost all math majors are. However, when they see enough disparate examples it all comes together, maybe weeks down the road. Give them the reassurances I discuss in the section of this manual entitled, "Dealing with 'math anxious' students" (page 38).

Aloud in class: A5-8

Section 1.2. Order Matters.

Some students may not realize the importance of the *order* in which operations are carried out. If students could learn to respect the importance of order in one class day, the day would be very well spent.

This section reminds them of (teaches them) the algebraic conventions about the order in which arithmetic operations are carried out. Far too many students do not know the conventions. Instructors continually see this in calculus, where important parentheses are omitted. For example, "$x+5(x+2)$" may be erroneously used to represent "$(x+5)(x+2)$".

A calculator which uses the algebraic conventions (not all calculators do) can be used to nicely point this out in class. Somehow calculators are authoritative. Ask them to guess what the calculator will say if you punch in:

$3 + 4 \times 5 =$

in exactly that order. Those who guess "35" need help (or a better calculator). Similarly, there is a common disregard for proper use of parentheses when squaring expressions like "$2x$." Some get "$2x^2$."

Other common order errors include division problems:
(3 + 4)/(5 + 10)
square root problems
√(9 + 16)
and problems with negatives
$-5^2 = -25$, not 25, which is $(-5)^2$.

This section is a good one to illustrate the role of conventions and parentheses their importance in a context (arithmetic) in which they can see the difference, even if they have a weak background. Also, if they ever fail to use, or misuse, parentheses in the future, you can cite this section and remind them that they are making a significant error. *Order Matters!* (You will have many opportunities when it is appropriate to say this throughout the term.)

Have students pronounce some expressions. Correct pronunciation requires understanding the order.

Class question on “Left to right order": In which order is $7\text{-}4x^2$ evaluated? Is the order left to right? Why not?

Aloud in class: 1.2, many problems, especially A25-72 and perhaps A7-20.

Section 1.3. Reading Mathematics

In class give examples of *placeholders* in *patterns* that are used to express processes of arithmetic. Look at Section 1.4 to see how this continues. Problems like A3-A14 in Section 1.4 are excellent classroom examples.

This section and 1.4 discuss procedures of arithmetic. The point is **not** for the students to know how to *do* the procedures. I hope they can do them already. The point is to learn to *write* them. They are learning to read and write.

Section 1.4. Algebra and Arithmetic.

Algebra is about operations and order. This section shows how many calculation procedures of arithmetic actually use different orders or different operations than apparently expressed. Identities are used to express the alternative procedures actually used. It is important to note that we often do advanced operations by doing simpler operations in a different order. For example, 5 - 8 = -(8 - 5), where we learned to do 8 - 5 long before we learned to do 5 - 8. This section emphasizes the way Mathematics expresses a “problem-pattern" and a corresponding “solution-pattern" using placeholders. Problems like B5-28 are excellent classroom examples.

This section is about writing. It is not about doing. It is not even directly about knowing how to do (although you may need to remind them of that). Be sure your students realize they are supposed to *write* about procedures.

There is a great deal of material summarized in this section. You need not cover it all because the main lessons about mathematical symbolism expressing thoughts about operations and order are emphasized again and again.

Aloud in class: Many, marked with ☺.

Section 1.5. Reconsidering Numbers.

The number line is an extremely useful tool. Unfortunately, not all students are comfortable with it, so it is worth some time.

The subsection, "Revising Concepts" (p. 57) is very important. To realize that we care about *truth*, students must also see statements which are false. Also, they must realize that in algebra we deal with *real* numbers, not just the counting numbers. Here are some good examples (because they are false but may appear true at first glance):

1) $ab = ac \Rightarrow b = c$ [for all a, b, and c]
2) $3x < 5x$ [for all x]
3) $x^2 \geq x$ [for all x]

These examples fool some students because they are so used to the counting numbers that they treat "number" and "positive integer" as synonyms. So, I have noticed that it takes a while for some students to think to look at zero (1) and negative numbers (2) as possible counterexamples. Also, many do not think of fractions (3). They need to be aware of real numbers. They do not need to know the actual mathematical definition of a real number–the idea that decimals such as appear on calculators are included in the real numbers is sufficient here.

This section contains some interesting theorems about inequalities that are just the right degree of complexity. They are not trivial; students frequently solve inequalities incorrectly. On the other hand, they are not too difficult. They serve as excellent illustrations of the use of logical connectives and placeholders.

Multiplying an inequality by a negative number is tricky (T1.5.3B), and solving inequalities with absolute values is not trivial (T1.5.5). Have the students learn Theorems 1, 3, and 5, and learn them by name, not number.

Terms: "number line" (with two interpretations of numbers), "location" and "directed distance," "less than," "greater than," "inequality," "absolute value," "conjecture," "counterexample."

Class Involvement: Get them to say statements **aloud**. If they know how to say them they will be more comfortable with the results.

Theorem 1.5.3 on multiplying both sides of an inequality by a number is a good one because the students need to pay attention to the hypothesis. Theorem 1.5.5 on absolute values supplements this nicely. You can ask problems where the hypotheses are important and replacement plays a major role in the solution. Some students resist simply replacing "$|x| < c$" by "$-c < x < c$." But exposure to the pattern and the idea of replacement is very helpful for "math anxious" students. Emphasize the use of the connective "iff" in problem solutions.

Many students are not comfortable with absolute values. They need to be able to, at least, solve "$|x-4| < 5$."
Emphasize the use of "iff" in the solution:
$|x - 4| < 5$ iff $-5 < x - 4 < 5$ iff $-1 < x < 9$.

Expectations. The logical connectives are studied in Chapter 3. Students need not fully understand "if..., then..." and "iff" yet. Students do not need to fully understand generalizations yet, nor do they need to fully comprehend counterexamples. Many students cannot properly express their own theorems such as the ones requested in B73ff.

Class Conjecture: If $b < 4$, then $|2b| < 8$. [false, let $b = -5$.]
We will return to the idea of a *counterexample* many times.

Class Questions: Theorem 1.5.3 is stated with strict inequality. How can it be restated with equality included?

Theorem 1.5.1 is stated with strict inequality. How can it be restated with equality included?
How does Theorem 3 change if $c \leq d$? $b \geq 0$?
How does Theorem 5 change if $|x| \leq c$?
How does Theorem 5 change if $-c \leq x \leq c$?

Use class questions to begin to shift the responsibility for truth to the students (and away from you, the authority). We want them to know, and be confident of, what is true. To attain that, they must begin to assume some responsibility. You, the instructor, can help them by asking them in class to guess if conjectures are true or false.

Aloud in class: Very many. [There are almost always many problems that could profitably be done aloud in class.]

Chapter 2. Sets, Functions, and Algebra. Chapter 2 introduces the vocabulary and concepts of sets, functions, and equation-solving. Equally importantly, Chapter 2 uses the basic vocabulary of Mathematics including the connectives "if and only if", "if..., then...", "and", "or", and "not". Pedagogically, in addition to providing concept images, this chapter provides formal concept definitions (for example, of set "intersection") which will be important later. Feel free to discuss any of these ideas more thoroughly at this time; however, connectives and definitions will be emphasized in Chapters 3 and 4.

Sets and functions are mathematical objects that are more abstract than numbers. Similarly, operations on equations (such as "Square both sides!") are more abstract than operations on numbers. Many of the activities of mathematics require attention to be focused on objects that are more abstract than numbers. For example, word problems (Section 2.4) utilize a functional level of abstraction where the students must focus more on relationship between numbers than on the numbers themselves.

Section 2.1. Sets.

This section is long and can take several days. I give it three. Some students need to learn content about sets, interval notation, and set operations at this stage. But this section also has lessons about connectives ("and", "or", "not", "if...then...", and "iff") as well as sets. The text will continue to discuss sets, and the students will see the formal definitions of the set-theory operations and relations again. The discussion of "and" in Chapter 3 helps students understand "intersection," and vice versa.

Lesson: "set", "member," "element," "equal," defining property notation, set-builder notation, interval notation, "empty set," "subset," "intersection," "union," "or," "complement," "universal set."

"Set" is undefined. So is set membership. Most students will not even notice that.

The definitions of "set equality" and "intersection" and "union" are given three times, in English and in two dialects of Mathematics. Students need to get used to the fact that the English definition is <u>not</u> the formulation in which mathematics proves results. English may help their understanding, but the mathematical formulation is the useful one. In pedagogical terminology, we want them to develop not only a concept *image*, but also a concept *definition* (page 257 in Section 4.5).

Note various versions of definitions. Note correspondence between set-theory terms and the logical connectives. Remark that we will eventually emphasize the versions with logical connectives.

Definition 4.5.1 describes the most useful mathematical form of definitions. We define a term by defining an entire statement containing the term. The definition is a second statement in more primitive terms. They are connected with "If and only if," the mathematical connective permitting substitution of one statement for another.

<u>Restricted Content</u>. Of course, sets can be very general and contain elements such as "Winston Churchill", but such sets rarely appear in mathematics. We limit sets to

sets of numbers. Therefore, we use lower-case letters for elements and reserve upper-case letters for sets.

In most math courses sets of numbers are the only frequently encountered sets. Almost all significant mathematical-language concepts can be discussed with sets of numbers. The empty set is then the only tricky concept. The idea of a set of sets can come later. Yes, it belongs to the subject of sets and is important, but it is not widely applicable at this low level. I would avoid it until later. If the students understand sets of numbers, intersection, union, complementation, equality, and subset, that is enough.

<u>Two Languages</u>. Students are happy to distinguish between English and Mathematics, the language of mathematics. This section begins doing so. It is important that there are two major dialects in Mathematics: one emphasizes notation and vocabulary, the other emphasizes the use of logical connectives. The former is more commonly used to **state** results (e.g. Definition 2.1.9B, page 76), the latter to **prove** results (2.1.9C). I use the term *translate* to refer to restating results in the other language or dialect. Note that true understanding of math will require understanding of logical connectives and generalizations, which are the subjects of Chapters 3 and 4. Be patient.

"Subset" is a great term. "$S \subset$ T" iff "$x \in S \Rightarrow x \in T$." This "sentence-form" definition uses the key connective, "$\Rightarrow$". I must use this definition 30 times during the term. Note that **sets are determined by their members**, so the mathematical form of the definitions of set-theory terms is in some sense "best", since it clearly shows facts about the members. For example, "$x \in S \cap T$ iff $x \in S$ and $x \in$ T," tells you precisely what is in "$S \cap T$."

The English version is good for understanding, but it is not the formulation in which the facts can be used. Section 4.5 "Reading Theorems and Definitions" is devoted to this idea. It does not need to be grasped by the students this early in the text (which is even before "if..., then..." is defined), but some indication of its application is appropriate here.

<u>Pronunciation</u>. Interval notation is hard to read out loud. You might as well admit it. How do you pronounce "$(a, b]$"?

To permit interval notation to be read out loud you might wish to introduce (but not define) the terms <u>open</u> and <u>closed</u> in connection with intervals. Let "a" and "b" be finite. Then
(a,b), $(-\infty,b)$, and (a,∞) are open.
$[a,b]$, $(-\infty,b]$, and $[a,-\infty)$ are closed. The last two, with the use of "(" instead of "[" when the endpoint is "∞", are minor problems. $(-\infty,\infty)$ is both open and closed, but that should not be mentioned. It can be read "the interval from $-\infty$ to ∞" because the "endpoints" are not real numbers and are therefore not included. I don't recommend explaining how an interval can be both open and closed at the same time! You can eat up class time with this if you go beyond the two key cases, (a,b) and $[a,b]$.

Then "(3,5)" can be read “the open interval three to five.”
"[4,7.1]" can be read “the closed interval 4 to 7.1.”
"(2,5]" is still hard to read out loud.

Some students may not pay much attention to the difference between {2,3}, (2,3), and [2,3]. After all, its only a minor bracket variation. But you might tell them about the zookeeper who said “I shot the bear in his cage.” He got a far different reaction than he expected, even though his sentence differed by only one letter from the intended “I shut the bear in his cage.” Little things do matter!

Many students find images of such sets drawn on the real number line to be illuminating.

Class Question: Why can't a set contain an element more than once? Answer: A set is determined by its members, not how many times they are listed, according to the *definition* of set equality.

There is such a mathematical concept (multiset) but it is not called a “set.” You can define anything. We happen to be studying a concept in which one of each element is all we want. For instance, if I ask “What is the set of solutions to $x^2 = 9$?” the answer is {-3, 3}, not {-3, 3, 3, 3}. We want each one listed only once.

Question: Why can't (or aren't) the elements of a set be ordered? Answer: There is such a mathematical concept, but it's not called a set. It's a matter of the definition. We are not studying ordered things.

Aloud in class: A5-14, 21-24*, A30-64, B19-22

Section 2.2. Functions.

This section deals only with the most important types of functions, real-valued functions on the real numbers. Furthermore, for low-level students, the key concept is that a real-valued function is a *rule* of correspondence between elements of the domain and real numbers. This correctly relates declarative-mode symbolism with students' natural imperative-mode thinking. At this level there is no need to go into sets of ordered pairs or functions that have unusual domains or images such as “Winston Churchill.” There is no need to deal with functions in full generality now.

Many students have seen only the "$y = x^2$" type of notation for functions. In calculus, the "$f(x) = x^2$" approach to functions is dominant. The advantages of using "f" are significant.

This is a good place to again note the role of placeholders. This is discussed at length in Chapter 4, but should be introduced now. For example, “Let $f(x) = 3x$, for all x,” defines the same function as “Let $f(y) = 3y$, for all y,” and describes $f(x+h)$ too.

Class Questions. I think the students can benefit from translating their own definitions into mathematical notation. For instance, suppose you wish to describe a function which automatically doubles any given number. How would a mathematician write it? "$f(x) = 2x$."

Class Question: A furniture retailer has the following pricing policy. Everything is priced by doubling the wholesale cost and then adding $10. How would a mathematician express that relationship (between wholesale and retail price)?

Student Errors. Composition of functions is a non-trivial concept closely related to the concept of order. At the beginning many students do not recognize "$f(x)$" as describing an operation (rather than just a number). They cannot correctly compose such functions until they can read the operation, or sequence of operations, from the notation. Also, they may not recognize the order in "$f(g(x))$".

Expectations. Many will not understand placeholders at this stage. They are still focusing on the numbers and not on the properties of the operations. This is an excellent subject to help them. Do not expect it to be easy. They will see this again, very thoroughly, in Chapter 4.

New Vocabulary. You could introduce "f is an increasing function" (Example 3.4.14, p. 191 and Example 4.5.5, page 255) if you would like another "if...,then..." term to utilize in examples.

Section 2.3. Solving Equations.

This material could take several days to cover. Please do not regard this section as merely a "how to solve equations" section. Not all problems are algebraic, but even in the ones that are **the solution itself is secondary to the process** and the recognition of exactly what is going on and why.

Tell your students that all the rules are listed together at the end of the section, page 128.

Students should learn

1) that solving equations is a logical process
2) the connection between logic and solving equations
   (especially the connection between "$\Rightarrow$" and extraneous solutions)
3) how to solve equations (at least in these few cases)
4) how Mathematics expresses processes
5) operations on equations (e.g. "Square both sides") produce related equations, and which operations produce which relations is important.

This section teaches the justification for a process (equation solving). Many students have almost learned how to solve equations for unknowns, but too often drop or add solutions along the way. They will learn precisely what can be done about that, and they will practice doing it, step by step. Many other students can solve equations with fair success, but don't understand why the steps work, or don't work. Both groups of students will learn a lot about algebra in this section. For maximum benefit, they should be given a small or moderate number of problems to **analyze**, rather than a large number of problems to solve without analysis.

They can learn how not to drop solutions and they can learn what steps may introduce extraneous solutions and what to do about it.

They will study the proofs later in Chapter 5. They are excellent illustrations of the role of logic in proof (Chapter 3), because the rules follow by logical manipulation from the three simplest rules. It is good to see and try to understand connectives (as in this section) before actually concentrating on "proof." By the way, by being treated in detail, solving equations can prepare the student for the sort of careful analysis required in proofs.

Homework problems are easy to create for this section. There are many long problems. I do not think students need to do anywhere near all of them and I don't think the students should do many per night in any case. If too many problems are assigned, students rush to get the answers–missing the emphasis on analysis. A few problems done in detail will illustrate the point far better than a homework assignment which is so long the student must fall back on old sloppy ways just to get enough answers. Answers are not the subject-- methodology and its understanding are.

I have clearly asked in the homework problems that students explicitly note the use of the rules and state the resulting implication. The point is partly to bring the use of rules to the conscious stage and to train the students to have citeable reasons for their steps. These are important in proofs.

In this section there are really only three primary rules, Rules 1 through 3. The rest are logical manipulations of these. There are other possibilities for rules along the same lines that I have not stated which could be discovered by writing the forms of these rules and trying the possible equivalents, using the ideas in Chapter 3. There are advanced homework problems requesting more "rules."

Thus this is a great subject for studying proofs.

Example that picks up an extraneous solution: $x+1 = \sqrt{(x+13)} \Rightarrow (x+1)^2 = x+13 \Leftrightarrow x^2+2x+1=x+13 \Leftrightarrow x^2+x-12 \Leftrightarrow (x-3)(x+4) \Leftrightarrow x = 3$ or -4. But in the original equation $x+1$ would be negative if x were -4.

Class Research Question: Give a complete result about the solutions to $ax = b$.
i) if $a \neq 0$, then $x = b/a$; ii) if $a = 0 = b$, then all real numbers solve the equation; iii) if $a = 0$ and $b \neq 0$, there are no solutions.

Near the end of the time on this section, you might ask them to try Problems C5-8 about stating and proving new theorems. They are ready (almost eager) for it, and learning to state theorems is part of the goal.

Additional problems: Show all steps, all implications, and cite reasons for every step.

1. Solve $(x+5)(x-2) = 2x(x+5)$. [-5 or -2]
2. Solve $(x-2)/x = 2x-4$. [2 or 1/2]
3. Solve $4 + \sqrt{(2x)} = x$. (Rule 7 yields an extraneous solution) [$x = 8$] [2 is extraneous]

Aloud in class: A1-6*, many others.

Section 2.4. Word Problems.

Some word problems are closer to arithmetic than algebra. This section makes the distinction and shows that in algebraic word problems formulas that are conceptually functions play a critical role. This should connect well with Section 2.2 on functions.

Because the hard part of word problems is setting up the equation to be solved, which depends upon building the relevant formula, this section emphasizes only obtaining the relevant equation. I do not emphasize obtaining the solution, which is a number. I avoid emphasizing numbers in this context because the difficulty of word problems is conceptually at the level of functions, not numbers.

Expectations: Some students cannot even do direct word problems. I'm sorry, but this section is not addressed to them. It is over their heads and they will not get it. Use this section to help that small percentage of the class that can be helped in just two days. Do not expect miracles.

Aloud in class: none

Chapter 3. Logic for Mathematics.

Chapter 3 begins the subject of logic <u>for mathematics</u>. Logic for mathematics is only a small fraction of the logic in a basic logic course. We emphasize only logical equivalences and deduction (using truth tables as a tool, not as an end). Chapter 4 emphasizes the use of the quantifiers "for all" and "there exists."

I have the students flag pages 199-201 where the logical equivalences are all listed together

The logical equivalences are selected because they exhibit the most common patterns of mathematical thought. A list of examples of important mathematical sentences that illustrate these patterns is given below.

Students find the material in Chapter 3 on truth tables complete and satisfying. They are able to construct truth tables easily, but many do not have an easy time remembering or reasoning with patterns with 3 variables (such as Theorems 3.2.9 and 3.3.6, pages 199-200).

Chapters 3 and 4 thoroughly discuss **truth**, a concept without which the declarative (fact-based) language of mathematics cannot make sense. Section 4.6 summarizes the main ideas. Look at Section 4.6 to see where the text is headed.

In logic we study form, not meaning. Form is abstract, and therefore most students are much more comfortable with studying meaningful sentences with terms they understand than studying (meaningless) forms. Nevertheless, the emphasis is on the patterns, not on the examples of the patterns.

Most students can correctly create truth tables of new statement forms. This is an area where even the weakest students usually have success. Whether they truly grasp the use and importance of the forms is a different matter. Forms are abstract and some students can reason out the relation between two particular sentences in a way they cannot yet do with the forms. The instructor's method of helping solve this problem is, of course, to repeatedly go back and forth between particular sentences and abstract forms (See the sample sentences below under "Examples of Logical Patterns"). It helps to manipulate sentences using the equivalences cataloged as "important logical equivalences." The goal is to get the students to recognize the role of form alone. This subject continues through the end of the text.

Eventually, advanced mathematics students must become absolutely comfortable with the logical equivalences. Students might, at first, regard the theorems as merely exercises in the creation of truth tables, but they are far more than that. These thought patterns should become so familiar that they become "natural." They should be memorized by name (Of course, the theorem numbers are not important). However, many weaker students do not attain the level where they can reason with three abstract variables.

Student Difficulties. Students have trouble seeing the three (3) sentences in "$H \Rightarrow C$". They may say "It is true" and you may not be able to tell which "it" they have in mind. I get them to say "the conclusion is true" or "the conditional sentence is true," whichever they really mean. Too many students focus on the conclusion. For example, if they are told "$H \Rightarrow C$" is true, some will assert that therefore "C" is true, without any thought for "H". You can correct this misconception, but it takes time and many repetitions.

"It" is a word that gets a lot of class time. Students soon learn not to say "it" in the context of conditional sentences when they realize how ambiguous it is.

Thinking at the (Abstract) Pattern Level. I have found that many "math anxious" students need to go over the same examples of equivalent (or not equivalent) sentences again and again. Fortunately, even the good students seem to benefit from this. Perhaps it is a bit like basketball, where even the pros have to practice shooting each day.

Good Examples of Logical Patterns. It is helpful to have a few key examples in mind which you can use again and again. Here is a list of examples which can be manipulated using the patterns in the "important logical equivalences" of Chapter 3. You can illustrate any of the basic types of logical equivalence using these examples.

Theorem 1.5.1: $a < b$ iff $a + c < b + c$.
Theorem 1.5.3B: If $b < 0$, then $c < d$ iff $bc > bd$.
[The reversal of the direction of the inequality helps make the hypothesis "real" to them.]
Theorem 1.5.5 (on absolute values): $|x| < c$ iff $-c < x < c$.
["$-c < x < c$" abbreviates "$-c < x$ and $x < c$," and its good for them to become aware of abbreviations.]
Definition 2.1.8: S is a subset of T iff $x \in S \Rightarrow x \in T$.
Definition 2.1.9C: $x \in S \cap T$ iff $x \in S$ and $x \in T$.
Definition 2.1.10C: $x \in S \cup T$ iff $x \in S$ or $x \in T$.
Definition 2.1.20C: $x \in S^c$ iff $x \notin S$ (and $x \in U$).
Definition 2.1.4C: $S = T$ iff $x \in S$ iff $x \in T$.
Example 14, Section 3.4, p. 191 and Example 4.5.5, page 255:
f is increasing iff $x < z \Rightarrow f(x) \leq f(z)$.

$x > 5 \Rightarrow x^2 > 25$	true
$x^2 > 25 \Rightarrow x > 5$	false, because of a counterexample
$x \leq 5 \Rightarrow x^2 \leq 25$	false
$0 \leq x \leq 5 \Rightarrow x^2 \leq 25$	true
$0 \leq x$ and $x^2 > 25 \Rightarrow x > 5$	true
$x > 5 \Rightarrow 3x > 10$	true, even when $x = 3$ or 4

$|x| \geq c$ iff $x \leq -c$ or $x \geq c$.
[split "iff" theorems in two by Theorem 3.2.5]

$x > 5$ or $x < -5 \Rightarrow |x| > 25$	[LE on cases, T3.2.8]
$|x| > 25 \Rightarrow x > 5$ or $x < -5$	["or" in the conclusion, T3.3.5]

$S \subset S \cup T$.
$S \cap T \subset S$
$ab = 0$ iff $a = 0$ or $b = 0$
$ab = 0 \Rightarrow a = 0$ or $b = 0$
$a = 0$ or $b = 0 \Rightarrow ab = 0$.

$ac = bc \Rightarrow a = b$	false
$ac = bc \Rightarrow a = b$ or $c = 0$	true

I have the students flag pages 199-201 where the logical equivalences are listed together.

<u>Comments</u>: "not" and "and" do not cause much trouble. "or" is tricky in the context of solving equations. Is the solution to "$x^2 = 4$" given by "$x = 2$ and $x = -2$" or by "$x = 2$ or $x = -2$"? It is also not easy for some students to distinguish "A and B implies

C" from "*A* or *B* implies *C*." The connective "iff" is pretty easy for them. The really tricky connective is "if.., then...." To understand the truth-table definition, they respond best to the "broken promise analogy" (p. 148), but their understanding is more of social responsibility than the connective itself. Example 14 (p. 149) gives a mathematical reason why "$\Rightarrow$" is defined the way it is. This reason relates closely to the algebra in Section 2.3, but students do not seem to relate to it well. It will be covered again in Chapter 4.

<u>The Difficulty Level of the Logical Equivalences</u>. Referring to the summary list of logical equivalences on pages 199-201.

T3.1.14 No one has difficulty with double negation.
T3.1.12 "$H \Rightarrow C$" is LE to "(not H) or C" is useful but not intuitive. Most students have a hard time remembering this.
T3.2.2 Most get the contrapositive without much trouble.
T3.2.4 It only takes a few examples in class before the converse is easily distinguished from the original statement.
T3.2.5 They easily understand the "Theorem on `iff'."
T3.2.7 "Two conclusions" is easy.
T3.2.8 "A hypothesis in the conclusion" is extremely important for higher math, and not too natural. It is very useful in proofs. For example, to prove that, under some hypothesis, "$S \subset T$", you are likely to translate "$S \subset T$" into "If $x \in S$, then $x \in T$" and then you will have a hypothesis ("$x \in S$") in the conclusion. Most such proofs use the rearrangement of the LE.
T3.2.9 "Cases" mixes "and" and "or" in a way they do not find natural.
T3.3.2 "Negation of a conditional sentence" is **the worst**. They have an intuitive feeling for a "counterexample," but they cannot see the *pattern* in the concept. You will often see them write that the negation of "$H \Rightarrow C$" is "$H \Rightarrow$ not C", or some other incorrect permutation. In fact, **some students repeatedly get this wrong**.

This is the most dramatic instance of students being able to do the right thing in particular examples and not be able to give an abstract formulation of what they did. That is, they can say why "$x^2 > 100 \Rightarrow x > 10$" is false, but they don't see it abstractly. It is false because there is a case where the hypothesis is true **and** the conclusion is false.

T3.3.3 DeMorgan's laws they may not know at first, but they pick them up easily. The switching of "or" and "and" is something they can do.
T3.3.4 "A version of the contrapositive" is very useful in high-level classes. They have some trouble remembering it. For example, "$0 < x$ and $x^2 > 25 \Rightarrow x > 5$" is true, where as "$x^2 > 25 \Rightarrow x > 5$" is not. "$x < 5 \Rightarrow x^2 < 25$" is false. By this theorem, "$0 < x$ and $x < 5 \Rightarrow x^2 < 25$," which is true.
T3.3.5 "or" is easy to explain

T3.3.6 "or" in the conclusion. The example (half of the Zero Product Rule): "$ab = 0 \Rightarrow a = 0$ or $b = 0$" is important (it's an important justification for factoring). $|x| > 3 \Rightarrow x < -3$ or $x > 3$.

T3.3.8 I do not emphasize "proof by contradiction" at this stage. Many proofs said to be "by contradiction" are really "by contrapositive" anyway, so T3.2.2 usually suffices. Section 5.6 covers the distinction.

T3.3.10 Tautologies are easy, but Theorem 3.3.10 is hard for them. I included it because many logic texts avoid stating logical equivalences. Instead, they state long tautologies corresponding to my LE's by using the alternative form justified by Theorem 3.3.10. The alternative form is far less natural and does not reflect the way mathematicians or students really think. The approach I take, which is to state logical facts as logical equivalences, is far more effective.

T3.3.11 This substitution process is natural for them. They would do it happily without reference to this theorem.

T3.3.12 "iff conclusions" fits the form of the theorem on multiplying the sides of an inequality by a number, Theorem 1.5.3. It is important for higher-level math–not so important here, expect perhaps as an example where the substitution justified by Theorem 3.3.11 is natural. That is, students now know enough to create new logical equivalences by working with results about truth tables rather than by creating more truth tables. This is a nice advance in reasoning skill.

T3.3.13 In class, you might wish to draw a parallel between the Distributive Properties of logic (T3.3.13) and the Distributive Properties (Property 1.4.9) of arithmetic: $a(b + c) = ab + ac$, where "and" is like multiplication and "or" is like addition. That may help, but they still have a bit of trouble remembering them.

T3.4.1 Transitivity of "$\Rightarrow$" and "iff" are natural.

T3.4.3 Modus Ponens is obvious to them.
The placement of "and" and "or" in T3.4.3B and T3.4.3C does cause some trouble.

T3.4.3D is obvious to them.

T3.4.4 They love Theorem 3.4.4! They find that it explains a lot about the organization of proofs, which most have found a mystery. They grasp it right away.

The patterns from Chapter 3 are used repeatedly in Chapter 4. If the students do not have all the logical equivalences at their command, they will pick them up in later chapters. As an instructor, you want them to learn the theorems by name and to be able to state them correctly, but they will have much more practice with them in significant contexts later, so students who have trouble with patterns can still learn them later.

By the end of Chapter 3 they are much more aware of the importance of *patterns*. But most do not yet think at the pattern level. Their thinking is not yet that abstract. But, they are much closer.

Appendix: Logical Terms Worth Omitting

Unfortunately, many results in logic have more than one name, and most of the names from a logic course are not illuminating. Furthermore, most of the usual names are not used (or even known) by regular mathematicians. Thus I have chosen to use illuminating names, keeping the students and mathematical practice in mind. For my own curiosity I have compiled a list of alternative names which is given here. I recommend you omit them. The terms I do not use are unnecessary and generally not illuminating.

Note: In logic, a “horseshoe” ($\supset$) is a symbol for the mathematical "$\Rightarrow$". This is very unfortunate for Mathematics, because the horseshoe points the wrong way for the correspondence between subset ($\subset$) and "if..., then...” (p. 84, 155). I haven't yet had any students comment on this, and I certainly do not comment on it (Why confuse them?) but those who have taken logic might find this reversal confusing at first.

The term “hypothesis” is similar in Mathematics and in logic, but not quite identical. In logic, a “hypothesis” is a conjecture offered as a possible explanation. In Mathematics, a “hypothesis” is the "A" part in a conditional sentence of form "$A \Rightarrow B$." If the conditional is true, the hypothesis may serve as an “explanation,” but it would not be just a “conjecture” as a “possible explanation,” rather a sufficient condition.

text usage	alternative omitted usages
"A and B"	the conjunction of A and B
"A or B"	the disjunction of A and B, alternation, inclusive disjunction
"$A \Rightarrow B$"	implication [this term is misleading, because in English the “implication” is the conclusion, not the whole sentence]
"A iff B"	biconditional, equivalence [this is misleading without variables. How can 2+2 = 4 be equivalent to the Fundamental Theorem of Calculus? Both are true, so "2+2 = 4 iff FFT" is true, but no real “equivalence” is expressed.]
conditional sentence	implication, conditional
hypothesis	antecedent, premise
conclusion	consequent
contradiction	absurdity

transitivity of "⇒"	law of syllogism (in logic, a "syllogism" is any statement with two premises and one conclusion
Modus Ponens	law of detachment, law of the excluded middle
contrapositive	transposition
proof by contrapositive	Modus Tollens, indirect proof, law of contraposition
proof by contradiction	law of conjunctive inference law of conjunctive simplification reductio ad absurdum, indirect proof
open sentence	propositional function, predicate

Chapter 3 is about the "Propositional calculus" and Chapter 4 about the "Predicate calculus." Neither term is very illuminating.

variable	indeterminant
T3.4.3	law of simplification
T3.4.4	law of addition

Section 3.1. Connectives. Section 3.1 is pretty typical for the subject matter, except that I avoid non-mathematical examples. Of course, good examples from mathematics will be generalizations (the subject of Chapter 4), so not much math can be done at this stage. But that's fine. Students are quite happy to just do truth tables and learn logical equivalences. Even the worst students can do it and it gets them thinking that maybe they can do math after all. There is no need to seek too many realistic examples at this stage. However, there is good reason to avoid silly examples.

The only tricky connective is "If..., then...." It is traditional in logic courses to give silly examples of "⇒".

Bad Example: "5 > 3 implies Los Angeles is in California" is true.

Technically, this example is correct. If the hypothesis is true and the conclusion is true the conditional sentence is true. But, it certainly does not illuminate the meaning of "⇒" and only serves to confuse the students. I have very carefully avoided giving examples of true simple conditional sentences. I have found that most examples do harm rather than good (unless they have variables, like my examples).

Inferior Example 2: If you cheat, I will quit playing. I do not quit playing. What can you deduce?

Research shows that humans have innate social abilities that serve to deal with such examples, regardless of logical abilities. Students like such examples, because they do not have to develop logical abilities to handle them. The lesson, if there is one, is not math. Unfortunately, examples like this do not help students learn logic. The real reason why "$\Rightarrow$" is defined as it is cannot be understood until sentences *with the same variable in the hypothesis and conclusion* are connected by "$\Rightarrow$". Then there are lots of good examples which use variables. Use some of the text examples, such as "$x > 5 \Rightarrow 3x > 10$."

It is important for the students to realize that statements can be false. Open sentences, such as equations to be solved (e.g. "$2x + 5 = 13$") can be false. I put up a number of statements such as "5+2 = 7", "5(2) = 12", "4 > 6", and "5 < 9" and give them letters. Then the students seem to have an easy time agreeing on the definitions of "not", "and", and "or", which seem to parallel English. "If and only if" does not cause problems either.

Conjecture: "($A \Rightarrow B$ and B is false) $\Rightarrow A$ is false." [true]

Notation. There are many notations for "not." I use "not." You may find it convenient to introduce an alternative shorthand for boardwork and homework. For "and" I am equally comfortable with the whole word, "and," and the symbol, "$\wedge$", which is really only found in logic, but looks like a capital "A" and is therefore easy to learn. Later, when it comes time to state mathematical theorems, the word "and" is obviously preferable.

Advanced material: All the other connectives can be defined in terms of "and" and "not." Advanced students (only advanced student) could be asked to find a compound statement logically equivalent to "A or B" using "A" and "B" and using only the connectives "and" and "not." Ditto for "$A \Rightarrow B$" and "A iff B."

A or B is LE to not[(not A) and (not B)]

$A \Rightarrow B$ is LE to not[(A and (not B)]

This will become far easier after they have practiced more.

Aloud in class: many.

Section 3.2. Logical Equivalences.

The logical equivalences emphasized here are genuinely important because mathematical sentences are often reorganized using these logical forms. They provide "sentence synonyms" that every mathematician must know.

I emphasize logical equivalences where some authors state the corresponding tautologies (using Theorem 3.3.10). Tautologies are an important part of logic and they play a large role in proofs, but **I have found that the difficulties students have with proofs are not usually in misusing tautologies, but in not knowing the basic logical equivalences.**

You might tell the students that mathematicians think of logically equivalent statements as expressing the same meaning.

I state the basic tautologies used for deduction in Section 3.4. Of course, tautologies and logical equivalences are closely related, but by studying logical equivalences first we can help students recognize the synonyms of Mathematics. Thus Chapter 3 emphasizes the part of logic they can use to deal with individual sentences. The role of tautologies in proofs (paragraphs) is made clear in Chapter 5 on proof and introduced in Section 3.4. Their use is illustrated, but I think you will find that students do not need much instruction to use them properly.

Aloud in class: A5-11.

Section 3.3. Logical Equivalences with a Negation.

Negation of a conditional sentence is hard for them. They can do examples, but have trouble remembering the pattern. In spite of your best efforts, some students will repeatedly come up with incorrect negations of conditional sentences. This needs a lot of emphasis.

Students can **use** the concept of a counterexample correctly by now, but many will not be able to reproduce its abstract form: The negation of "$H \Rightarrow C$" is "H and not C" (Theorem 3.3.2). And, in the context of a generalization, “For all x, $H(x) \Rightarrow C(x)$” is false precisely when “There exists an x such that $H(x)$ is true and $C(x)$ is false” (Chapter 4.3). The *and* is **very** hard to remember. I emphasize is strongly in class and still many students wrongly insert "$\Rightarrow$" instead. By the way, I recommend you anticipate the “there exists” version (formally Theorem 4.3.4, page 230) so that you can explain why "$x^2 > 25 \Rightarrow x > 5$" is false. You need the “there exists” for counterexamples to make sense. This anticipation does not bother students.

I deemphasize “Proof by contradiction.” Many proofs said to be “by contradiction” are actually “by contrapositive,” so Theorem 3.2.2 on the contrapositive is actually used more. See Section 5.6 (which I don't reach) for the distinction. A second version is in Problem B25.

They cannot *read* T3.3.10 or 3.3.11. T3.3.11 they understand perfectly well, they just cannot get it by reading. Substitution of logically equivalent statements is very common in mathematics. In many math books, you will frequently see substitution used, completely without comment, and students are not bothered by it.

On the other hand, T3.3.10 is hard for them. I stated it because some logic texts use the tautology form to state facts which I think are more appropriately stated as logical equivalences. This way students can see the correspondence between what they state and what I state.

Aloud in class: A9-24, A26-28, 29-76. Many others.

Section 3.4. Tautologies and Proofs.

Most students have an easy time with tautologies. They love Theorem 3.4.4 (page 189) which explains a lot about proofs.

The preview with actual proofs and manipulation of forms is harder and need not be mastered at this time. It is good for the students to know where they are going. The exposure is important, but they need all of Chapter 4 before we can hope for mastery. However, this preview prepares them to expect to translate sentences (to exhibit the connectives) and rearrange sentences (into useful logically equivalent forms).

The idea that the capital letters we have used to represent statements may represent components of realistic mathematical sentences (Theorems) is important and could have been introduced earlier. But I like to let the students master the logic by itself before also requiring them to translate at the same time.

Aloud in class: few

Examples: Here is a nice group of "Modus Ponens" examples for class, which could be entirely omitted or deferred until Chapter 5 on proof.

Begin by asserting that you are thinking of a number (integer) and you will give the class two clues about it and then ask a question about it. The class is to use the clues to answer the question, if possible, and assert that there is not enough information, if not possible. The clues concern whether the number (positive integer) is greater than 30 or not and whether it is odd or even.

Example 1: I am thinking of an integer.
Clue 1: If it is greater than 30, it is even.
Clue 2: It is even.
Question: Is it greater than 30?

Example 2: I am thinking of an integer.
Clue 1: If it is less than or equal to 30, it is odd.
Clue 2: It is less than or equal to 30.
Question: Is it odd?

Example 3: I am thinking of an integer.
Clue 1: It is greater than 30.
Clue 2: If it is odd, it is less than or equal to 30.
Question: Is it odd?

One colleague of mine asks his students if they would be willing to "bet their right arms" on their opinions. Some say "Yes" and then prove to be wrong! It seems that many students blithely use any clues to "prove" anything without reference to how the clues relate logically to the alleged conclusion. A few exercises of this nature might help to shape them up.

I recommend having them extract the "form" of the clues and question to check for logical equivalence. Let the unconditional clue be "A." Rewrite the clue in conditional sentence form using "A" and "B", and identify the conclusion. Then check to see if the from is either Modus Ponens or its logical equivalent "[$A \wedge$(not $B \Rightarrow$ not A)] $\Rightarrow B$."

Example 1 (revisited): "It is even" is A. "It is greater than 30" is B. The hypotheses are "A" and "$B \Rightarrow A$." We cannot conclude "B".

A quick counterexample would be 16.

If you wish to defer such examples until later, there is another good opportunity to introduce them in Section 5.1.

Note that these conclusions are not mathematical facts; they are just facts about particular numbers you are thinking of.

Note to problems B3-8: To prove "H ⇒ C" is true, you only need rule out one of the four possibilities (the possibility that "H" is true and "C" is false).

Chapter 4. Sentences, Variables, and Connectives. The emphasis of Chapter 4 is on translating mathematical sentences into other mathematical sentences which appear different, but express the same meaning. Mathematicians do this all the time, but hardly notice the process since they are fluent in Mathematics. Section 4.6 summarizes much of the material. Look at it in advance to see where we are headed.

Students have seen some of Chapter 4 already because they have seen variables and connectives in action. Nevertheless, many find Chapter 4 a challenge. It seems harder than Chapter 3 because students must bring all of Chapters 1 through 3 to bear. The amount of mathematics is piling up. Chapter 4 asks students to recognize and master all the mathematical usages of variables. This mastery is a tremendous accomplishment.

This is the chapter in which most students, even weak ones, make the transition from resisting pattern recognition to grasping it. After they make the transition, many students recognize a distinct difference in their own thought patterns. They are conscious of a change, sometimes virtually over night. They come in telling everyone that they "get it," and that "It's simple." It *is* simple, but very *profound*, and not easy.

Students are happy to learn that there are 3 basic types of sentences with variables. I think they knew something was odd when similar sentences had different interpretations. Section 4.1 tells them they were right, and they are pleased.

The difference between an identity (as an open sentence which is always true) and its corresponding generalization is too subtle for them. It is similar to the difference between a set and its members. The question is, is it being regarded as one, or as many? Understanding the key difference depends upon an understanding of abstraction, and not all of them understand it yet. Basically, abstraction can turn the essence of many similar things into one thing. If "$2(x + 5) = 2x + 10$" is regarded as a separate sentence for each x, it is *many* (therefore, an open sentence which is always true). If it is regarded as an abbreviated generalization, it is *one* (therefore, a true generalization). I'm almost sorry they worry so much about the distinction, but they do, and it shows they are aware of subtle distinctions that had escaped them before. That's good.

They respond well to the idea that we do not need to distinguish between singular and plural (p. 216). Section 4.2 has material on dependent and independent variables that is not easy. Section 4.3 is straightforward, but new to them.

Section 4.4 points out how English can be ambiguous when Mathematics is not. They understand this.

They respond well to switching letters in the quadratic formula (Section 4.5, page 252-253). Many had a fear of that formula, and they are now capable of mastering it (and they know they can). Some used to think the key to the "a", "b" and "c" was the left-to-right pattern rather than the "multiple of x^2", "multiple of x" and "constant" pattern. They figure this out now with examples like homework A41 and B13-19 (p. 263, 265). Their pride shows.

Section 4.5 on theorems and definitions is hard. Here they must finally understand placeholders. There is a very simple idea behind definitions, but that idea is truly profound: *equivalence*, which permits replacement. We want them to take a sentence with a new term they don't understand and translate it into (replace it with) an equivalent sentence expressed in more primitive terms. Then they can deal with it in those more primitive terms. However simple this is in principle, it is not their natural approach and they need to be reminded that it is the thing to do.

Hardest is the idea of replacing a vague understanding of a term (“concept image”) with a formal “concept definition”(page 257 and Definition 4.5.1, page 254).

After we use *equivalence* to solve equations in Section 2.3 “Solving Equations,” they will have seen it in many contexts. A great question is “What is the role of *equivalence* in Mathematics?” They will know (because you go over it a lot), but they resist using it with definitions. I've thought that their reluctance is related to concerns about plagiarism! They've been told to put their ideas in their own words for years–now I tell them to use *my* words. To some that just seems wrong–like cheating.

Section 4.6, “Different Appearance–Same Meaning,” summarizes everything you can do to express mathematical thoughts in different manners. It pulls together all the ideas of Chapters 3 and 4.

Section 4.1. Sentences with One Variable.

This section pulls together a lot of what we have been talking about all term. Some terms are new (e.g. "existence statement”), but the thrust is consolidation and clarification of familiar ideas. Students are now sophisticated enough to be able to see why two sentences may look similar and yet be interpreted differently.

I tell them (admit to them) that a lot of mathematics is written in a manner which does not explicitly distinguish the difference between two quite different kinds of sentences. They can understand that by analogy with the following example which I put on the board: What is the meaning of the word "lead"? If they answer about leading and following, I say it refers to the metal, and if they answer about the metal, then I mention leading and following. With a smile I then note that it is obvious you need context to really know. Similarly with math. Sometimes you need context to know, and the context may be your own prior knowledge. They can deal with that.

They can better understand that it is their responsibility to decide what they are reading from the context when they realize that they do it all the time in English.

In Section 4.1, I have distinguished three meanings of "$f(x) = 2(x + 5)$." It could define f. If f is known, it could be an equation with an unknown, x, or it could be an identity with variable x.

Is it any wonder that mathematics is confusing to uninitiated students?

Student difficulties: The difference between the generalization "For all x, $2(x + 5) = 2x + 10$" and the identity "$2(x + 5) = 2x + 10$" is very subtle. The latter could be regarded, technically, as an open sentence that is always true. It is the extreme case of "open sentence." Therefore, we can (and almost always do) treat it as an abbreviated generalization. Both sentences tell us nothing about x, the number. This distinction is so subtle that I prefer not to go into it at length. It doesn't easily satisfy students.

I think it works well to do 4.1 and 4.2 together.

Aloud in class: A2-5,10-11, B4-10

Section 4.2. Generalizations and Variables.

The emphasis is on alternative appearances of generalizations. Some are explicit, but many are implicit. The students must be able to recognize implicit generalizations.

The comments about the (lack of) distinction between singular and plural are usually well-received by the students.

The role of generalizations in “true-false” questions is important, since it is fundamental to mathematical thought.

It is now time for students to realize what mathematical sentences are about. I hope by now they have the higher-level concept of operations and order, and realize that some sentences are about such high-level mental objects.

Parameters can be nicely introduced with some sequence of thoughts like this: Solve $2x + 1 = 7$. Solve $2x + 1 = 21$. Solve $2x + 1 = ...$, well, we are always doing the same steps, so we could make it "$2x + 1 = c$" and solve. Now "c" is a parameter describing a family of related equations. Now move to “Solve $3x + 1 = 19$,” and as many more similar problems as it takes. They should rapidly see the point. We could

"Solve $ax + b = c$" and have a family of equations with 3 parameters and derive the solution of all similar equations once and for all. "x" and "c" play different roles.

Aloud in class: A1-14, 18-25, B4-30. Refer to page 128 which has "Rules of Algebra" stated with parameters and ask which letters are parameters.

Section 4.3. Existence Statements.

The negation of a generalization is an existence statement. Some students will want to express the negation of "All are ..." with "All are not ...." This misconception must be addressed.

Example: Ask them to consider this statement: "All the people in this class are males." That is false, but not because all are females, rather because some are females. Mathematicians use the word "some" but the use of "there exists" is more precise and it is the way the theory is phrased.

"$x^2 = y^2 \Rightarrow x = y$" is false, but so is "$x^2 = y^2 \Rightarrow x \neq y$." This illustrates that the negation of a generalization is not a generalization.

Expectations: I would hope students could by now write the abstract form of the negation of "For all x, $H(x) \Rightarrow C(x)$." Nevertheless, there will be some who can use the concept of counterexample but still can not write the form "There exists x such that $H(x)$ *and* not($C(x)$)." The *and* is the problem. I guess some students do not think that abstractly. Oh well.

My term "positive form" (Definition 4.3.3, page 230) helps prepare the students for an important idea. We prove something false by proving its negation true. To do so we usually need the "positive form" of the statement in question.

Make sure they learn to disregard the normal inferences from singular and plural. In math, singular may refer to more than one, and plural may refer to only one. Class Question: Conjecture: There exist element<u>s</u> of {1,2,3} in {2,4,6,8,10}. Answer: Yes, since we disregard plurals.

<u>Student Difficulties</u>. Generalizations can have implicitly quantified variables, so some students fail to note that the negations have a "there exists". For example, the problem: State the negation of "$x^2 > 25 \Rightarrow x > 5$," might yield the answer, "$x^2 > 25$ and $x \leq 5$," with the quantifier incorrectly omitted.

A hard problem could be: Give the negation of "For all $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x-c| < \delta$ implies $|f(x) - L| < \varepsilon$." Even outstanding students commonly forget to quantify the "x" in the negation, since it is not explicit in the original sentence.

Aloud in class: A1-25

Section 4.4. Ways to State Generalizations.
They can do this.

The opening discussion about "or" is important.

Note how the plurals and the words *all* and *any* express generalizations, and the *has* and *is* may indicate existence statements.

Aloud in class: A1-6, B5-16.

Section 4.5. Reading Theorems and Definitions.
It is time for students to have mastered placeholders. Placeholders are a critical concept for reading theorems. Example 2, on the Quadratic Formula, is an excellent lesson. You can spend a long time on problems with increasing degrees of difficulty.

1) $3x^2 + 5x - 10 = 0$. 2) $4x^2 - 6x = 30$ 3) $2x^2 + kx - 8 = 0$.
4) $dx^2 + px + k = 0$. 5) $bx^2 + 2cx + d = 0$.
6) Solve for p: $2p^2 + dp = 20$. 7) Solve for y: $2xy + 3y^2 = 50$.

Point out how place (in an equation expressing operations) is the key idea, not the particular letter holding the place.

I like to put up problems like

Definition: $x@y = 2x - y$. Solve $20@x = (2@1)(3@2)$.

This takes a real understanding of placeholders.

Example 11 is a very illuminating example. The ability to replace something that is not understood, such as n^*, with something equivalent which is understood, is a major part of fluency in Mathematics. If you put up, in a pseudo-quiz atmosphere, "Definition: $n^* = n/2$ if n is even and $n^* = 3n$ if n is odd. Solve $n^* = (10^*)(5^*)$," many students will read the odd-looking equation to be solved and stop dead. When I wander by and offer aid, they may well ask, "What's n^*?"

Of course, that is the point of defining n^*. It tells you 10* and 5* and, even, n^*. If they would have the courage to REPLACE 10* with 10/2 = 5 (because 10 is even), they would be part way through. I think many students do not see "=" as permitting replacement. In many cases, it is just a symbol in the middle of a problem to do. The definition of n^* with its "=" sign does not seem to trigger the thought that we can replace the unfamiliar n^*, 10*, and 5* with more familiar expressions.

The key idea to reading definitions is that **Logic applies to sentences, not words**. Therefore, we define sentences containing vocabulary words, not just the words themselves. A word needs a context, which a sentence provides. Thus, when it comes time to apply logic to mathematical sentences containing vocabulary words (as in proofs), we do not replace just the vocabulary words with their definitions, rather we replace entire sentences with their defining conditions. Mathematical definitions in "sentence form" define entire sentences by sentences expressed in more primitive terms.

Student errors: Students may have difficulty distinguishing between quantified and free variables. For example, which letters can be changed in "If $x \in S$, then $x \in T$" without changing the meaning? Only "x". That sentence has the "for all x" suppressed. But "$b > 0$ and $c < d \Rightarrow bc < bd$" could have any or all the letters changed, since all are quantified.

Class Questions: Ask them to rewrite results and theorems they have seen using different letters. Results from Section 1.5 and Chapter 3 make good examples.

What is the contrapositive of "$C \Rightarrow$ (not B)"?

<u>Student Difficulties</u>. Students have trouble seeing the three (3) sentences in "A iff B". When they say "it" in class you need to discover which "it" they have in mind. Sometimes "it" is "A", sometimes "B", and sometimes "A iff B".

For example:

$S \subset T$ iff if $x \in S$, then $x \in T$.
the sentence being defined its definition
|----------------the definition in sentence-form -----------|

It is possible to use notation to give definitions that do not appear to define sentences. For example, consider Definition 2.1.9B (page 76) of set intersection: "$S \cap T = \{x | x \in S \text{ and } x \in T\}$." That definition is correct and fine, but, a set is determined by its members. So that is only abbreviated way of telling us when "$x \in S \cap T$" is true. That is what Definition 2.1.9C tells us. You can't prove anything about the intersection of sets without first exhibiting a sentence with the connectives displayed. If you think you can, it is because you are so fluent you don't even notice the translation. Students need to make the translation step explicit.

Students should be warned that many authors (even of high level texts) use "if" in definitions when the really mean "iff." For example, "An integer is even if it is divisible by 2." For another, "b is a bound of S if, for all $x \in S$, $|x| \leq 12$." Both of these are technically incomplete.

What not to memorize: It is not intended that they memorize what an "interior point" is (although some students erroneously think they must memorize it because it is in the text). It merely serves as a moderately complex term for them to use as an *example*. They are to learn *how* definitions state what they state, not just *what* they state. This section is about the requirements of definitions in general, not about particular definitions.

Class question: Give an equivalent form of "x irrational $\Rightarrow x$+8 is irrational." They may even be able to prove it, given the contrapositive form, and the definition of "rational."

Class Question: Give an equivalent form of "If x is rational and not 0 and y is irrational, then xy is irrational."
$A \wedge B \Rightarrow C$ is LE to [(not C)$\wedge A \Rightarrow$ not B], that is, if x is rational and not 0 and xy is rational, then y is rational.

Advanced Question: Ask the class to create a definition of "local maximum" giving them hints from pictures. Let them find the existence part.

Students must have the concept that 2 sentences can be equivalent ("sentence synonyms"), that is, say the same thing without appearing exactly the same. So, for 4.5, you must:

1) be willing to replace one sentence with another, equivalent, sentence
2) see unfamiliar terms in their entire context, that is, in a sentence.
   [We do not define words, but rather sentences]
3) If you are uncomfortable with a term, throw out the whole sentence with it and replace the sentence with another without the term.

Aloud in class: A5-8, A35-38

Section 4.6. Different Appearance–Same Meaning.

This section is a review section. It summarizes all the ways we have to rewrite an individual sentence without changing its meaning. Students should memorize the five major ways by name and be familiar with the minor ones too.

Point out that "equivalent" is not the same as "logically equivalent." Yes, "logically equivalent" implies "equivalent," but sentences can be equivalent for other reasons besides logical equivalence. Sentences that are logically equivalent are equivalent solely because of the arrangement of the components and connectives, not because of meaning.

When students have these mastered, they know all about mathematical sentences. Chapter 4 begins the discussion of how sentences can be strung together to make useful paragraphs. Some paragraphs solve equations and some yield proofs. Chapter 4 shows some of both types. Chapter 5 emphasizes proofs.

Student Difficulties: Some students memorize the names of the five ways, and memorize examples of the five ways, but don't get their examples to correspond to the right names. This is a sign of pure memorization without understanding. For example, there is a difference between a definition and a theorem that some students do not grasp.

Aloud in class: A1-14. Maybe B15-17.

Chapter 5. Proofs. Now students have assimilated enough preliminaries to understand *proof*. Without them, *proof* would be very hard to grasp. With them, even the previously "math-anxious" students are ready. The essential preliminaries include: truth-based mode of thought, use of language and especially placeholders, connectives and logic, patterns of reasoning, justification, and concept definitions. All of these are discussed in Chapters 1 through 4.

The major ideas underlying the concept of proof are given in Sections 5.1 and 5.4. Section 5.2 is practice and emphasizes the importance of steps being prior results. It uses conjectures to force the students to think analytically. Section 5.3 emphasizes the role of formal definitions of terms. Sections 5.4 through 5.7 continue the subject, but the structure of proof is clear after Section 5.2. If a course only got through Chapter 5.2, it still could be considered a complete and well-rounded course, although students would not have had much practice with proofs.

Not many books, even books discussing how to do proofs, actually state what a "proof" is. I do. I want the students to be able to write a proof; it only seems fair to tell them what it is I am asking them to create.

Sections 5.1 and 5.4 cover the entire theory. The key to concept of proof is its two parts:

tautological form and *prior results*.

They have been studying *form* explicitly ever since Chapter 3 (and implicitly since the beginning of Chapter 1). The new part is to get them to respect prior results. That is, to not use just any steps they want or like, but steps which are *justified by prior results*. The "list" approach (pp. 285ff) is helpful in this regard.

They find the reason a two-step proof "works" to be illuminating. They seem truly interested in learning the answer to the question, "What is a proof?" Most texts which claim to teach how to do proofs do not clearly explain what a proof *is*. Not only does this text explain what a proof is, most of the *students* can explain what a proof is after Section 5.2.

They are curious about the difference between "prove" and "deduce" (p. 310).

Success and Pace. At this stage, which for my classes has been close to 36-40 class hours along, genuine mathematical thought processes are occurring to even the most "math anxious" students. And my classes experience very few drops among students who work. The drop out rate is nothing like the usual high rate of other math classes. That is because the students are brought along successfully. Anything they cannot do or grasp at first is eventually assimilated because everything is reused frequently.

The remaining sections in Chapter 5 reinforce all the lessons of 5.1 and 5.2. In 40 class periods you can get through Section 5.2 even with a high percentage of originally very weak students, if they work at it. Surprisingly, I was unable to go much faster when I taught this to school math teachers! It seems that this material can be studied at several levels, and previous success at *doing* math does not assure that the logical underpinnings have been mastered. Also, previous *lack* of success at *doing* math does not mean that the logical underpinnings will be hard to master. When the language is mastered, the *doing* makes sense, so students who have previously "failed" at math can and do succeed with this approach.

Section 5.0. Why Learn to do Proofs? This is just a short section worth less than a full lecture. I give a selection of answers to this question. Sometimes my students have come up with very good answers of their own. You may well have other equally cogent reasons.

Section 5.1. Proof. This section begins with a definition of *proof*. It discusses the requirements. It gives a few simple proofs. Finally, it distinguishes proofs from arguments.

If you use sets as a context, then you need to translate. Translation has been discussed throughout and is also the subject of Section 5.3. Students do not need to do difficult examples at this stage. They need to grasp the role of form, which is best illustrated with shorter examples. One difficulty with good math students is to get them to distinguish "true" and "prior." A sequence of **true** statements from which we can deduce the result we want is **not** a proof unless the steps are **prior** results.

One of the greatest difficulties in teaching how to do proofs is in specifying what the students already know and can use in their proofs. For example, if you are doing algebra and you start with laws on the level of the commutative law of multiplication ($ab = ba$) many will not even notice when they use it because they are very familiar with the properties of addition and multiplication of real numbers. Pedagogically, it is easier to start with more sophisticated results first. Then, if the student continues to be interested in the foundations of mathematics, he or she can learn more about the elementary properties by taking an abstract algebra course or a foundations-of-mathematics course.

The one idea that seems to help the students the most is the idea of a "list" of prior results. The "list" approach has several virtues. One, it is a mathematically correct way of describing how mathematicians regard "truth" (and "falsehood"). Two, it is a mathematically sound way of describing how the subject of mathematics is advanced. Three, it is a good way to get them to understand and play the game of doing proofs by employing only prior results. Yes, proof is a game with rule–we tell the student the rules.

This section leads smoothly into 5.2. Section 5.1 is short so you may wish to begin 5.2 before finishing 5.1, and, for that matter, you can bring in 5.3 as well.

Section 5.2. Proofs, Logic, and Absolute Values. The steps of a proof are supposed to be based on prior results, so the definition of "prior results" is important.

The "list" idea of Section 5.1 is important.

As a teacher, one difficulty is to find good **short** examples of proofs. Another is to find a topic which is simple, but not so simple that students are convinced they already know the results and therefore resist proving them. That is why this section uses absolute values. In some sense they are simple, but it is easy to state conjectures that look plausible but are not true. It is a worthy subject for proof.

I recommend you accept the spirit of the text and ask the students to work on the conjectures. There are enough theorems proved to show how proofs are done.

It may be time (again) to remind the students that they are not studying formulae or results, but the methods of derivation of the results. Thus, the result need not be interesting to have an interesting derivation. In fact, it must be very simple or the derivation is likely to have a very complex form–too complex to illustrate the points we are making.

Technically, long proofs would have long tautological forms. But I do not think that mathematicians actually have a long tautology in mind when they prove a substantial result. But they can see that the steps are conclusive. Most long proofs just use an extended version of transitivity of "if..., then...".

In class: Have the students go to the board in groups and work on Conjectures 5.2.12 -24.

Section 5.3. Translation and Organization.

The proper organization of proofs is discussed. Sometimes the form of the original statement is changed, and sometimes new terms have to be translated first. The point about logically equivalent forms being proved instead of the stated result is important. This is rarely covered properly in lower-level courses. Definitions in sentence-form are important and so are the logical equivalences of Chapter 3. If the translated form fits one of the logical equivalences in Chapter 3, proofs almost always use the alternative form.

Section 5.4. The Theory of Proofs.

Mathematics is a very concise language. It is not always easy to understand. Even if a paragraph of Mathematics contains only correct statements, it may well be hard to follow. There are at least three reasons for that. One is, the paragraph may begin without motivation, in which case the reader may feel lost before he gets started. Another is, the connection between the sentences may be unclear because they are not in the expected logical sequence. If each sentence does not follow from the one previous to it, the reader may not be able to keep track of what has been deduced and why. The third problem is that the reasons connecting the statements may been unstated and the reader needs to be reminded of some older result which permits the step. Another version of the same problem is that the steps may be too sophisticated for the reader's comprehension level.

The writer of Mathematics can and should do something to help the reader with each of these problems. This section is intended to give students some guidelines for writing readable proofs.

First we will consider the problem of motivation. Frankly, most mathematicians don't feel a strong need to motivate their sentences. If my objective is to prove "$H \Rightarrow C$", and the proof can be found in a sequence of statements, the fact that the sequence contains the proof is motivation enough for most mathematicians. After all, the proof and its discovery are not the same. If the point is to present a proof, the bare-bones proof is enough. Of course, if you are trying to **teach** the proof, then motivation is very helpful.

Discovering a proof is scratch work. The proof itself must simply insert prior results into a proper form; technically, it need not indicate how it was discovered. Of course, proofs that do not include some indication of how they were discovered often leave students unsatisfied.

Section 5.5. Existence Statements and Existence Proofs.

Existence statements are more common in higher-level math than in algebra. For example, in calculus, the concept of a limit of a function, which is an existence statement, plays a fundamental role. Here are some existence statements about functions that are related to limits.

For the conjectures let g be the function defined by g(x) = 7x.

Conjecture 1: There exists $x > 3$ such that $g(x) < 22$.

Conjecture 2: There exists a $d > 0$, such that $g(x) < 22$ if x is in $[3,3+d)$.

Proof: Let d =.1. Then x in $[3,3+d)$ implies $x < 3.1$ and $g(x) = 7x < 21.7$.

Conjecture 3: If $c > 0$, there exists a $d > 0$ such that $g(x) < 21+c$ if $x < 3+d$.

Proof: For $c > 0$, let $d = c/7 > 0$. Then $x < 3+d = 3+(c/7)$ implies
$g(x) = 7x < 7[3+(c/7)] = 21+c$.

Conjecture 4: For z, there exists y such that $g(x) > z$ if $x > y$.

Conjecture 5. Fix c. Then, for any $e > 0$, there exists $d > 0$ such that
$|f(x) - f(c)| < e$ if $|x - c| < d$.

Section 5.6. Proofs by Contradiction or Contrapositive.

Proofs by contradiction and proof by contrapositive are compared. You might note that many proofs said to be "by contradiction" are really "by contrapositive."

Section 5.7. Mathematical Induction.

Induction proofs are very difficult for students who have not been trained to respect the difference between a conditional sentence, its hypotheses, and its conclusion. The induction theorem has a conditional sentence as a hypothesis ("For all n, $S(n) \Rightarrow S(n+1)$"). To untrained students, the hypothesis of that conditional looks a lot like what they are trying to prove ("For all n, $S(n)$"). This is confusing.

Those two sentences are quite different. Furthermore, we do not assume "For all n, $S(n) \Rightarrow S(n+1)$" -- we prove it, in order to satisfy a hypothesis of the induction theorem. When the two hypotheses are shown to be satisfied, the desired conclusion is proved.

The text has more than enough examples. You can use homework problems, e.g. B4 or B13.

There are recursive definitions (B7-11) , for example the Fibonacci sequence in which each term is the sum of the previous two, and the sequence is initiated with 1,1. Thus, $a_{n+1} = a_n + a_{n-1}$. It is common to start the indexing with 0, although beginning with index 1 is also seen (in which case the sequence begins with 1,2,3, not 1,1,2,3). The terms are 1,1,2,3,5,8,13,21,34,....

Example which can be done two ways: $1+4+7+...(3n-2) = n(3n-1)/2$.
One way: The usual induction. Ask students for the general term before giving it.
Another way: After having done $1+2+3+...+n$, the above sequence is three times it minus a sequence of 2's.

Conjecture: $(1-a_1)(1-a_2)...(1-a_n) \geq 1-a_1-a_2-...-a_n$.
Example: $(.99)(.95) = .9405 > 1-.01-.05 = .94$. It holds in this case. Does it always hold?

Dealing with "Math Anxious" Students

I love teaching this course. Students who are already good at math learn a lot they didn't know, and students who are afraid of math become excited about math and often tell me this is their favorite course! However, math-anxious students do need some special treatment, and this section offers advice about how to deal with intelligent, but math-anxious, students.

Students learn a lot of math skills and gain comfort with, if not necessarily fluency in, symbolic Mathematics. For math-anxious students this is a very tall order. They come in feeling lost and, for them, success is not in sight. You need to convince them that "not in sight" is not equivalent to "impossible." Explain that comfort with mathematics is simply a long walk down a well-marked road with a few turns. Every step along the way is a minor success. Then, lead them along that road and encourage them by commenting favorably on the progress they make, even if consists of steps that may seem "minor" to you.

The course is designed so that students can appreciate that they are taking steps in the right direction long before the end of the road is in sight. Furthermore, it is designed to begin at the beginning so that everyone can start on the right road. Many students have fundamental misunderstandings of algebraic notation and would never make progress if they were allowed to try to build on their foundation of sand. They must begin over.

Obviously, the math-anxious students need to learn things that math majors have already mastered. Because you, the instructor, are already fluent, you might not realize the very basic nature of the difficulties that math-anxious students have.

Many math-anxious students:

- cannot read symbolic mathematical symbols aloud
  e.g. cannot pronounce "$3(x + 5) = 21$," or "$2 < x < 5$" or "$|x|$" or "$\{x \mid x^2 \leq 9\}$," even after they have been studied.
- have little sense of precision with symbols
  e.g. They may square "$2x$" and write $2x^2$ instead of $(2x)^2$, or multiply $x + 2$ by 3 and get $3x + 2$, neglecting symbols that express order of operations. They may use "<" when "≤" is correct. After studying sets, they still might not distinguish between $(2, 5)$, $[2, 5]$, and $\{2, 5\}$.
- have difficulties with replacement
  They may simply not see errors in replacement, e.g. In a sequence of equations to solve, they may replace "$f(x)$" with "x" without reason. They may replace $(x + 3)^2$ with $x + 3$. (I think these prompted by the desire to "simplify".)
  They may decline to do replacement when it is appropriate, e.g Given the theorem, "$|x| < c$ iff $-c < x < c$," and the problem "Solve $|x - 4| < 1$," they may first add four to both sides to (erroneously) get "$|x| < 5$."

- have difficulty recognizing that symbols may hold positions
  (i.e. difficulty with placeholders. Is this related to dyslexia?)
  e.g. Given the definition "Let $f(x) = x(x + 1)$," and repeated instructions to focus on the positions held by x, they may be unable to follow the instruction "Substitute x^2 for x everywhere" to get "$f(x^2) = x^2(x^2 + 1)$."

Psychology is important. Many math-anxious students

- are very afraid of symbols, often to the point of paralysis. There reaction to math may be FEAR.
- are baffled at the beginning of every new method, every new notation, and every new topic
- expect failure with every new topic, regardless of successes with the previous topics
- do not expect that they can possibly learn math by reading
- strongly resist following instructions
  e.g. If you give them "Let $f(x) = x^2$" and ask them for $f(y)$ they may not know what it is. That's OK. They are learning. But, suppose you tell them, "Write what I tell you! Write exactly the same thing I wrote in quotation marks except replace every x with y." Many will stay paralyzed and write nothing; they cannot follow instructions when the context is the dreaded symbolic Mathematics.
- firmly reject a step-by-step approach
  e.g. If asked to create a truth table with columns for each appropriate combination, they will skip combinations or abbreviate so much they make mistakes. If asked to solve an equation and cite the reason for each step, they will avoid a new equation for each step and make up their own (erroneous) steps and omit the requested reasons. If instructed how to do a multi-stage problem with a sequence of steps, they may leap to an incorrect answer, avoiding the carefully outlined steps, or, alternatively, be paralyzed and write nothing. (Actually, I think many are unwilling (= unable) to write anything because they regard steps as totally worthless -- to them only the answer is worth anything and it's too far away.) Some will even explain sophisticated right brain/left brain reasons they cannot follow instructions that take several steps.
- have difficulty with replacement
  e.g. Suppose $f(x)$ is given. The problem "Solve $f(x) = 12$" may suddenly yield "$x = 12$" and the student does not notice the difference. Or, given "$\sqrt{x} = 4$" the next line may be "$x = 4$." This appears to be a lack of being careful, or a lack of precision, but it may be something more. It is very hard to cure, even with students who give evidence of

> being very careful in areas other than math. I think there is something affective about math symbols that impedes certain students who are terrified of math. I do not think this would happen to young people being first exposed to symbols, but it happens with students-over-traditional-age who have had math failures in the past.

The cure. There is a cure to most of these problems, but it requires going back to basics. The knowledge of algebra that math-anxious students have is a foundation of sand. They cannot build on it. They must start over and rebuild the foundation by using symbols properly in simple contexts. You, the instructor, must begin with fundamentals such as pronunciation of symbols, order of operations, and basic thoughts about arithmetic expressed in symbols.

Here are some very strong recommendations:

- Make them pronounce Mathematics aloud.
  It is hard to read if you don't know how the symbols are pronounced. My informal research shows that many student learn mathematical results as sequences of *sounds*. They do not necessarily remember results visually. Therefore, they must know how to pronounce them or they can't learn!
- Make them participate in answering simple questions in class.
  After all, this is a language class. They must learn to communicate. If the questions are very simple, even math-anxious students can build confidence, as well as use their abilities.
- Make them participate in noticing operations and using symbols.
  E.g. Write "12 - (-5) =" on the board and ask someone to evaluate it. Then say, "How did you do it?" Then ask for the problem-pattern: "$a - (-b)$." Then ask for the solution-pattern and the entire identity which expresses the method used: "$a - (-b) = a + b$."
- Write conjectures on the board and have students judge whether they are true or false.
  This requires them to read with precision and to take responsibility for truth and falsehood. E.g. After studying absolute values in Section 1.5, write, "Conjecture: If $a < b$, then $|a| < |b|$," and ask "True or false?"

<u>**The cure at the psychological level**</u>. You must realize that math-anxious students may be taking great strides forward merely by learning things you and good math students do naturally. They need encouragement and positive feedback for any successes, even successes that may seem minor, such as pronouncing "$2 < x \leq 5$" correctly.

Some students will do well immediately. But some will not. You need to encourage these students when they are baffled, because **many will be baffled at first and yet do very well later, after they "catch on."**

Here are some useful things to say repeatedly:

- This is a foreign language.
- This is Japanese! (or, Greek. Pick your favorite.)
- When you begin to study German, you don't expect to understand it all right away. You are given sentences to translate word by word that are hard to read. You don't expect to be good right away. But you do expect to get better, and by the end the sentences you had trouble with at the beginning will seem really easy.
- You will be graded primarily on the results of exams on Chapters 3, 4, and the comprehensive final (which includes part of Chapter 5). We never drop any topics, so you will get better and better at them. You will be best near the end of the term, when you will have seen all this many times. Therefore, your grade will be based primarily on your work near the end of the term. Every exam covers all previous material. The final is comprehensive. If you are good at the end, you will get a good grade, regardless of what you did not understand now. Just keep at it.
- You will get this. Every kid in Japan can speak Japanese. You just have to be around it and try.
- You don't have to understand it all now. We will go over this again, many times.
- We never drop any topic. If you don't get it now, just keep working at it.
- This is Japanese and you are just beginning and you are not yet fluent! But, we will be studying this same thing again and again and you will have lots of chances to see it again and you will become fluent. You don't need to be good yet. No one would expect you to speak Japanese well after a few weeks of Japanese class. My requirement is only that you speak it well *near the end of the course.*

This is a course in a foreign language, with odd symbols interpreted in non-intuitive ways. I liken it to Japanese. In class I even call it "Japanese" (many times)!

Students who have previously done poorly in math are afraid of math. They are afraid of every new section and every new topic -- even if they did well on all the previous sections. One semester is not long enough to convince them emotionally that THEY CAN DO IT. They expect math to consist of failures. Or, at best, short rote procedures that can simply be memorized and executed. But this course asks for true understanding and that takes longer. FEAR. "I'm completely lost!" You must reassure such students that "We are talking about difficult concepts **you will get**, but they take a while to grasp. That's why we will cover them again next class and in all the later chapters. We never drop any topic. You will get more fluent as we use the same ideas again and again. You do not have to understand them today -- just by the end of the course."

"The Language of Mathematics" is taken by many students who are excellent in subjects far removed from science and mathematics. These people are not dumb. But, they are often confused and afraid. They do not expect success at math. You must work hard to change their expectations. However, success does not come by memorizing short procedures (as it can in high-school math). Therefore, you must help them realize that we are not teaching short procedures, we are teaching *understanding*. They have many satisfying occasions when they "get it." And, when they do get it, they really do understand it! Help them savor these experiences of success. Help them learn to expect minor successes with individual concepts soon, if not immediately.

Do not misunderstand me. Students love this class, especially students who are "math anxious." They are able to experience *success* and *understanding*, two motivations for work that they did not experience in their previous math courses. But they do need some tender loving care. Provide it, and you will be very pleased with the results.

The Language of Mathematics as a Core Course

Language facilitates thought. The purpose of *The Language of Mathematics* is to help students learn to read, write, and think mathematical thoughts in the abstract, symbolic, language of mathematics.

Most math courses emphasize **what** is said; this one emphasizes **how** it is said.

Mathematical results are expressed in a foreign language. Like other languages, it has its own pronunciation, grammar, syntax, vocabulary, pronunciation, word order, synonyms, negations, conventions, idioms, abbreviations, sentence structure, and paragraph structure. It has certain language features unparalleled in other languages, such as representation (for example, when "x" is a placeholder it may hold the place of any real number or any numerical expression, even "$2x - 1$" or "b"). The language also includes a large component of logic. *The Language of Mathematics* emphasizes all these features of the language (Esty, 1992).

Obtaining numerical answers is not the focus of the text or homework. Students learn how to justify steps, state (simple) theorems, analyze conjectures, use placeholders, reorganize statements logically, appreciate the importance of hypotheses, read theorems and symbolic definitions of new terms (which is hard for juniors in Advanced Calculus!), and begin to grasp the concept of *proof*. Together, these components add up to a thorough discussion of how to read, write, speak, and think mathematics.

Fortunately, mathematical sentences and paragraphs are generally written in a limited number of easily distinguishable patterns. Students who are taught to recognize these patterns find mathematics far more comprehensible than those who are not. Furthermore, their abilities to solve problems and do proofs are much enhanced (Esty and Teppo, 1994).

Most examples come from algebra, functions, and set theory (not trig or calculus), but the material is the language itself, which is essential for all areas of mathematics. Since this material is not emphasized in any other course, the course level is hard to peg. Some parts look like a “transition to advanced mathematics” course, but, with this unique approach, many students who regard themselves as “terribly math anxious” do very well with the material (Esty and Teppo, 1994).

References. For a thorough explanation of how the language is essential to mathematics, see “Language Concepts of Mathematics” (Esty) in *FOCUS on Learning Problems in Mathematics* 14.4 (Fall, 1992) pp. 31-54. For the effectiveness of this course, see “A General-Education Course Emphasizing Mathematical Language and Reasoning” (Esty and Teppo) in *FOCUS on Learning Problems in Mathematics* 16.1 (Winter, 1994) pp. 13-35. For an article on grading in the context of this course, see the *Mathematics Teacher*, 85.8 (Nov. 1992) pp.616-618 “Grade assignment based on progressive improvement” (Esty and Teppo). (This was reprinted in *Emphasis on Assessment*, NCTM, 1996, and was posted on the web by the Eisenhower National Clearinghouse For Mathematics and Science Education: www.enc.org/reform/journals/ENC2202/2202.htm

Word problems are not a major thrust of the text, but do receive one section. “Algebraic thinking, language, and word problems,” by Esty and Teppo (*Communication in Mathematics, K-12 and Beyond*, NCTM Yearbook, 1996) discusses the critical role of symbolic language in word problems.

Audience. Because the organization and emphasis of the material is radically new, the use of the text is not yet widespread. It is used at Idaho State, Montana State, Baker University, and the University of Missouri Kansas City. Some have decided that it will be required for Elementary Education majors. (It was not designed for them, but they seem to have special difficulties with abstract symbolism and this course can cure that.) At Montana State it has been successfully offered since 1988 to general students and every second summer to secondary math teachers (who, of course, knew the procedures of mathematics, but were not so comfortable with expressing them symbolically). The course was actually designed with freshman math majors in mind, but, general-education students in it found that they could “finally” understand mathematics, so, when the word got around, they became the vast majority of the audience.

Equivalent courses. Probably no other text yields an equivalent course. The level would be about equivalent to a basic logic course–but, it is only partly logic and, in *The Language of Mathematics*, the logic is illustrated by and selected *for mathematics*. The course is more sophisticated than Algebra II or Liberal Arts Mathematics, but the content is not at all like College Algebra or Precalculus. Surprisingly, many students who fail algebra (even remedial Algebra I) in college do very well in this course (if they are mature enough to read well and do the work). It counts as a “core” course in mathematics at the schools where it is used.