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UNDERSTANDING MATHEMATICS USING GRAPHICS CALCULATORS

Warren W. Esty
Professor, Department of Mathematical Sciences
Montana State University, Bozeman, MT 59717
westy@math.montana.edu

What makes a graphics calculator exercise a good one? How can we use an exercise to teach students something worth knowing, instead of merely exercising the student's calculator? What sort of exam problems would test the students on what they learned, as opposed to testing the power of their calculators?

To answer these questions, we must first decide what we really want students to know about algebra, now that calculators are able to do much of what we used to teach students to do. In only a few years graphics calculators have evolved into equation solvers and symbolic manipulators. Soon they will be able to scan in problems -- even word problems -- and simply give the answer. Will this eliminate the need to teach algebra? I think not, but I expect it will radically change the parts we emphasize. Computational aspects will become less important and conceptual aspects relatively more important.

Have we as educators and the public come to grips with the parallel concern about arithmetic? Calculators that can multiply and divide have been available for decades, but there is still no consensus about how important it is to teach long division. Calculators make the computational aspects of arithmetic less important, but it has not proved easy to describe an alternative curriculum that educators and the public agree is more valuable.

Similarly, it will not prove easy to change the algebra curriculum to accommodate this potential change in emphasis due to graphics calculators and symbolic manipulators. There is no consensus about what can be de-emphasized, what should be emphasized instead, what concepts need to be developed, or how to develop these concepts better with the aid of calculators. Textbooks used to all be clones of one another because, prior to the advent of calculators, algebra "by hand" had been canonized, but now the field is wide open.

Graphs have been and will remain an important part of algebra. This paper addresses the algebraic concepts that calculators can help develop. Some of these concepts relate primarily to graphs, but others are not necessarily graphical, for example, domain and range, solving equations, maximizing or minimizing expressions, and composition of functions. New types of homework and exam problems are given to illustrate how conceptual development can be promoted and tested. Problems are taken from *Precalculus Concepts* by Warren W. Esty and used with permission.

Most of the learning is done using graphics calculators. Even the testing can use graphics calculators, but, there is a way to minimize the impact of automatic equation-solvers and window-adjusters. In the examples below, note the use of arbitrary graphs and abstract questions (questions without particular numbers or graphs). Problems marked with a dot, •, require a calculator. Those marked with a pointing hand, ☛, are particularly suitable for conceptual development and testing because they do not require calculators.

Elementary concepts include vertical and horizontal lines and points.

☛ Give the equation of the a) x-axis. b) y-axis.

☛ Give the equation of the a) vertical line through $(2, \pi)$.
b) horizontal line through (-1, -2).

☛ If you sketch a graph with both on it, which is located first?
The line $x = a$ or the point (a, b)?

☛ Figure 1 gives (a, b) and (c, d).
Find the coordinates of points P and Q.

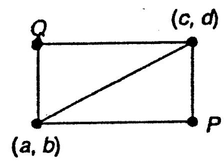

Figure 1

A "representative" graph is, by definition, not misleading and indicates the general behavior of the expression even for points not in the window (Esty, 1997, p. 52). Representative graphs can be used to "evaluate" expressions and "solve" equations. The relation of these as inverse processes is critical to most of the activities of algebra, including solving equations and doing word problems (Esty and Teppo, 1996).

☛ Figure 2 gives a representative graph of $f(x)$ without giving its algebraic expression.
a) Find y for $x = 2$. b) Find x such that $y = 3$
c) Use the "$f(\ )$" notation and the terms "solve" and "evaluate" to rephrase the questions in parts (a) and (b).
[(a) "Evaluate $f(2)$." (b) "Solve $f(x) = 3$."]

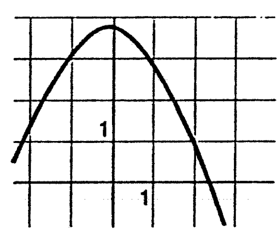

Figure 2

☛ Figure 2 gives a representative graph of f.
Use it to solve: a) $f(x) = 1$. b) $f(x) < 2$.

Evaluating the expression $f(x)$ for $x = a$ is accomplished graphically by reading from x ($= a$) to y. Solving the equation "$f(x) = b$" is accomplished by reading from y ($= b$) to x. These are inverse processes. Graphically, they are almost equally easy. The only difference is that there may be more than one x associated with a particular y, but, if $f(x)$ is given functionally, there is only one y for each given x. Until they grasp this, students may be satisfied with one x value as a solution in examples where there should be two or more solutions.

Graphically, evaluating and solving are similar, but algebraically they are much different. Expressions express how to evaluate themselves, but equations do not express how to solve themselves. There is no mystery to evaluating "$x^2 - 4x - 6$" -- the notation says what to do. But solving the equation "$x^2 - 4x - 6 = 0$" is much different. The notation mentions squaring, but the algebraic solution process does not require anything to be squared. Algebra emphasizes such indirect problems where the notation expresses operations that you are not supposed to do (Teppo and Esty, 1995). However, graphics calculators can solve equations using a direct method in which the operations expressed are executed in a "trial-and-error"-style procedure.

The *trace* feature of graphics calculators exhibits (x, y) pairs and illustrates the concept of a function as a set of ordered pairs. It illuminates the meaning of "solving" and distinguishes graphical solving from algebraic solving.

- Solve "$x + \ln x = 2$" graphically using the *trace* feature.

☛ a) To solve the equation "$f(x) = 0$" graphically, which expression or equation would we graph? [Graph the expression "$f(x)$" or the equation "$y = f(x)$".]
b) There is an important difference between the types of equations we solve and the types we graph. What is it? [The ones we graph have **two** variables, such as "$y = 3x + 7$." The ones we solve have one, such as "$12 = 3x + 7$."]

- Find the x-value that yields the maximum value of $6x + \sqrt{x} - x^2$.

Calculators can help explain the concepts of *domain* and *range*, which are related to the concept of *window*. Graphs of classic problems are illuminating.

- Suppose a farmer wants to make a rectangular animal pen with an existing long straight fence for one side and 100 feet of new fencing and gates for the other three sides. What dimensions of the pen would yield the maximum possible area of the pen?

First the relevant formula must be created. If "x" is the length of the side projecting from the existing fence, the formula is $A = x(100 - 2x)$. Graphing this, the standard window produces a useless picture. The concept of domain (as opposed to "natural" domain) is relevant to picking an x-interval. If the x-interval is adjusted first, then the *trace* feature can be used to determine corresponding y-values. The range and chosen y-interval are related.

- Solve $10 \ln x - x = 0$.

[There are 2 solutions, which is not obvious in the standard window!]

- Sometimes the "standard" window is not the one we want.

a) Sketch, in a window you draw, a "representative" graph of $y = (x - 20)(x - 50)$.
b) Give the x and y-intervals you used.
c) Find the minimum value of y for $x > 0$ in your window. [Great accuracy is not required, just get it close enough to show you know how to do it.]

• Figure 3 pictures the graph of $x(x - 20)(200 - x)$ in a certain window.
a) Find the window.
b) Find the coordinates of P.

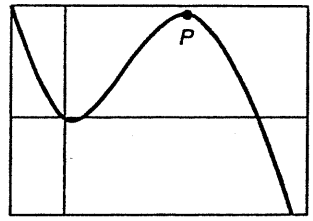

Figure 3

• Look at the graph of x^2 on [-10, 10] by [-10, 10]. Which new y-interval would make the graph exit the window at the upper right corner?

• Look at the graph of $y = x$ on [-10, 10] by [-10, 10]. Then change the scale to make it look steeper. a) Which x-interval makes it look steeper, [-5, 5] or [-20, 20] ?
b) Which y-interval makes it look steeper, [-5, 5] or [-20, 20] ?

☛ The figure marks some (arbitrary) points on the standard scale, [-10, 10] by [-10, 10]. Mark where they would be in the given windows (and note if they would be outside the window): a) [-5, 5] by [-10, 10]. b) [-40, 40] by [-40, 40].

☛ The (arbitrary) graph is in the window [-10, 10] by [-10, 10]. Sketch it in the window [-5, 5] by [-5, 5].

☛ Which y-interval, [-5, 5] or [-20, 20], makes graphs look taller?

Graphics calculators are wonderful, but they are not perfect. They may produce artifacts.

• a) Look at the graph of $1/(x - 1.5)$ on your graphics calculator. Does it produce an artifact? What is the artifact, if any? [Mention the model of calculator you use.]

Composition and decomposition of certain functions can be illustrated with graphs.

☛ Compare the graph of the given expression to the graph of $f(x)$:
a) $f(x) + c$ b) $2f(x)$ c) $f(x - c)$ d) $-f(x)$, etc.

The shapes of graphs determine the possible number of solutions to equations.

☛ How can you tell from the graph of a quadratic expression $f(x)$ how many real-valued solutions the equation "$f(x) = 0$" has?

☛ How many solutions can the cubic equation "$f(x) = c$" have?

Factoring, which has been a large part of Algebra I, can be facilitated with graphics calculators. The Factor Theorem comes alive.

- With the aid of a graph (and the Factor Theorem), factor these expressions:
  - $x^2 - x - 12$.
  - $3x^2 + 6x - 144$.

☛ How can its graph help you find the factors of a quadratic expression?

In calculus, graphs are frequently on a "square" scale so the slope is easy to estimate. However, many graphs use non-square scales.

☛ The slope of a line in a window depends upon the scale. Estimate the slope of the given line after noting the scale of the graph.

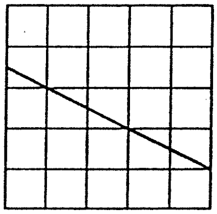

Figure 4: [0, 10] by [0, 1000]

Conclusion. These problems demonstrate that work with graphics calculators can facilitate the development of essential concepts of algebra in ways that were hardly possible before graphs were easy to draw and redraw. Furthermore, it is still possible to test the students rather than their calculators by including abstract questions and problems with arbitrary graphs.

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