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# LEARNING PRECALCULUS CONCEPTS USING GRAPHING CALCULATORS AND EMPHASIZING SYMBOLIC LANGUAGE

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Learning requires remembering. If students do not *remember* how to do something, they cannot claim to know how to do it. This simple fact has important implications for how to teach mathematics and how to involve calculators in teaching and learning. We need some emphasis on tools and exercises specifically intended to help students remember. Lessons can be designed to focus attention on generalizations and patterns. Pattern-recognition skills can help students summarize forgettable examples into memorable lessons.

Many students do homework by looking at the problems first, and reading the book only when they are stuck. Actually, "reading" may be too generous a term – many really just search for a worked example that they can parallel step-by-step. Or, they may look at a friend's work, copy it, and think "I see how that's done." Although the goal of a math assignment is to help the student abstract and learn methods which are applicable to many similar problems, the student's goal is often simply to "get the answer." Many homework problems are "done" without much focus on, or occurrence of, learning and remembering.

Furthermore, calculators are expressly designed to do many traditional problems and the sophistication of the types of problems they can do is increasing rapidly. Students with calculators may "do" many problems simply by knowing the right keystrokes, without learning the relevant mathematics we are supposedly teaching.

Given that there are these efficient (from the student's point of view) ways to "do" mathematics without learning it, what can we, as teachers, do to encourage students to focus more on *learning* and *remembering* and less on merely *doing*? What can we do to promote long-term learning that will allow students to recall and use methods in subsequent courses?

There is no easy answer to that. If there were, it would have been discovered long ago. It has often been noted that, at any level in high school and above, about half the students do not advance to the next mathematics course. And many of those who do take the next course amaze their instructors with their apparent ignorance of methods they have surely studied. It seems that all the examples the teachers worked and all the homework the students did somehow failed to cause the retention the students need.

One response has been to increase emphasis on real-world problems, sometimes including data gathering and modeling. The change is significant because texts from years ago largely neglected the real world in favor of pure mathematics. Another promising change is the introduction of calculators. In addition, this paper advocates increased emphasis on a another previously neglected area – the symbolic language of mathematics used to express patterns. The idea is that the symbolic language is designed to express mathematical thoughts and summarize mathematical methods, and therefore mastery of the language can promote mathematical thinking and facilitate retention.

When students need to solve an equation, we don't expect students to have seen that exact equation before, but we do expect them to recognize the *type* of equation, and to know how to solve that type. Recognizing a type requires recognizing a pattern – a pattern of operations and order represented by symbols. For example,

$$7x = 5 + 3x$$

can be solved by recognizing that one should "consolidate like terms." However, after studying logarithms, to solve the equation "$1.03^t = (1.5)1.02^t$" the equation

$$t\log(1.03) = \log(1.5) + t\log(1.02)$$

arises and, unfortunately, many students do not know what to do next because they do not recognize the pattern as identical to that of "$7x = 5 + 3x$."

Pattern recognition is the key to much of mathematics. The design of computer-generated tests illustrates that we want students to recognize patterns. For example, if we want students to know how to solve "$7x = 5 + 3x$" we must really want them to know how to solve "$ax = b + cx$," where $a$, $b$, and $c$ are the parameters of an entire class of equations. The test writer enters the pattern "$ax = b + cx$," and instructs the computer to generate values for $a$, $b$, and $c$ (so that $a \neq c$ and $a$, $b$, and $c$ are not 0). Then each student sees a specific example, say, "Solve $9x = 5 + 6x$.". The writer tells the computer the correct solution in terms of the parameters, $x = b/(a - c)$, and then the computer can check the student's answer, regardless of which problem of that type was generated.

Students know what we want them to know about that whole category of problems if they recognize the pattern and can solve the equation in terms of the parameters. So why not require them to do it? The method can be written symbolically as a theorem. Theorems are concise and precise ways of expressing what to do and when to do it. Solving "$ax = b + cx$" with "$x = b/(a - c)$" is *what to do*, and the conditions we had to program into the computer (such as $a \neq c$) to make the problem's numbers always work out right tell us *when to do it*. We want students to know the sorts of things theorems are capable of saying.

<u>Theorem</u>: If $a \neq c$ [and, for *this* "type," $b \neq 0$ and $a \neq 0$ and $c \neq 0$], then the solution to "$ax = b + cx$" is given by "$x = b/(a - c)$."

Mathematical methods are learned by doing numerous examples and generalizing. The general method itself can be summarized, once and for all, and communicated in written form, using abstract, symbolic, patterns. For students, writing a theorem about a method is far different from using the method several times on homework. The benefit is that students who are required to write summary patterns will simultaneously learn the patterns and how to read and write mathematics. Theorems in texts are supposed to be meaningful, but they are part of a symbolic language which is largely foreign to our students, even though the language is critical to algebra and all higher mathematics. When asked, many students admit they can't, or at least don't, read their texts. When a student says "The text is hard to read," there is more than one possible interpretation. The text might be poorly written, but my suspicion is that the student really means "I can't read math." Many students might add, "I never did learn math by reading. No one ever asked me to or taught me to."

This response is not surprising because courses do not focus on learning to read math. Most teachers do not expect students to learn by reading their texts; they have given up that fight. This being the case, learning outside class is virtually equated with "doing homework," which, experience shows, often does not yield the long-term learning students need. We might like to think of a text as a good lecture students can read at their own pace, but most never get that lecture. They can't read!

What can a course and calculators do to reduce the numbers of students who
1) do problems without remembering how,
2) use calculators to do math without learning it,
3) do not recognize and respond appropriately to various types of problems (patterns),
4) do not learn by reading outside of class, and
5) are unable to do word problems?

In the 1996 ICTCM Keynote speech Jacob Schwartz noted the two primary pedagogical virtues of calculators were that 1) they can increase the rate at which students can gather experience with the subject, and 2) they can concentrate attention on essential points. Experience focused on essential points leads to learning and remembering. Essential points are, by nature, abstract. If lessons are well-designed to lead to some "essential point," then students are expected to draw general conclusions about when to use various methods and how to use them – that is, to recognize and respond appropriately to various types of problems. The "type" of problem is defined by a pattern in the precisely same way a computer testing program generates specific examples from a pattern.

Patterns summarize learning. Students do problems without remembering how because they do not focus on, or recognize, the essential pattern. They are not explicitly required to focus on the pattern. Even if the text abstracts the pattern into a theorem, students are rarely held responsible for stating the theorems. We don't hold them responsible for knowing the patterns we want them to know! Could this be why they don't know them?

Does your course have any student activities that are designed to promote pattern recognition and reading skills? Homework can include problems with parameters and problems that require students to state – in modern mathematical symbolism – theorems that express methods. Does your course have activities specifically designed to help students generalize, abstract, and read at the conceptual level of theorems?

In his list of pedagogical virtues of calculators, Schwartz did not include that the TI-92 can do many sophisticated problems. Maple can do even more. Fine. But seeing neat graphs, or having a machine do problems, is not equivalent to learning. Montana has as much emphasis on calculators in the high school math curriculum as any other state, but the incoming freshmen do not *remember* their mathematics. If the current extensive use of calculators is supposed to be *pedagogically* effective, why don't students remember their math?

I submit that the ineffectiveness is closely tied to the insufficient emphasis on abstracting explicit patterns, which, in turn, is tied to lack of emphasis on using the symbolic language to communicate about mathematics. Without a strong grasp of symbolic language with which to think and remember, students have no way to summarize their learning. Many related examples do not coalesce into one simple concept. When the details fade from memory, as they surely will, the "essential point" of the lesson fades too. Students lack the language tools and skills that enable students to remember general methods regardless of details.

Among all the incremental changes that might help students learn mathematics, additional emphasis on mastering symbolism is appropriate. Calculators can help students see and do numerous examples of some sort of problem, and they can be asked to summarize what they have learned by writing the common method using the language of mathematics. Then they will have been forced to try to abstract the point of the many examples at least once.

The essence of algebra is its focus on operations and order (as opposed to arithmetic's focus on numbers). Graphing calculators can illuminate the concept of order.

Lesson. Graph $x^2$. Have the students predict the appearance of $x^2 + 4$ and $(x+4)^2$.
Graph sin $x$. Have the students predict the appearance of 2sin $x$ and sin($2x$).
Order matters! Have students generalize. Calculators can produce illuminating examples so rapidly that there is time enough to devote some of it to summarizing the lesson. Notation is critical. The ideas of using "$f(x)$" for a general function and some parameter, say $c$, may occur to only a fraction of the students. Given this helpful notation, they can write about the relationships between the graphs of $f(x)$, $f(x)+c$, $f(x+c)$, $c\,f(x)$, $f(cx)$, etc.

Lesson: Graph several quadratics of the form $(x - a)(x - b)$ with the intention of having the students note what they can predict about such graphs even before hitting the "graph" button. You know, and we can hope they will soon note, that graph crosses the $x$-axis at $x = a$ and at $x = b$. Next multiply out a few such examples and graph them using the unfactored expression. Use these examples to lead students toward the statement of a theorem about graph-aided factoring of unfactored expressions. Again, calculators can produce so many examples so rapidly that time is freed up to formally summarize the lesson.

Students are not used to writing theorems, and stating this one is hard for my precalculus students. They get ideas, but usually can not write them properly without a lot of help. Here are possible symbolic answers – versions of the well-known Factor Theorem:
"If the graph of $x^2 + cx + d$ crosses the $x$-axis at $a$ and $b$, then $x^2 + cx + d = (x - a)(x - b)$."
"If a quadratic with leading coefficient 1 crosses the $x$-axis at $a$ and $b$, then it factors into $(x - a)(x - b)$."
"If a quadratic crosses the $x$-axis at $a$ and $b$, then it factors into $k(x - a)(x - b)$, for some $k$."

My criterion for final approval of an answer is if it is stated clearly, concisely, completely, and correctly, as if stated in a mathematics text. If all a student writes is "$(x - a)(x - b)$", the answer is woefully incomplete. In class it is hard to resist treating such vagueness as exhibiting the main idea and only trivially wrong, but completing the statement requires clarity of thought, and the lesson should be designed to bring out *their* clarity of thought, not just show yours. The lesson should *teach* how theorems are written, not just show a correct theorem. Few beginning students have any idea of how to use parameters to state general truths in patterns. They need, but lack, this ability to abstract. It is not coincidental that the corresponding methods are grasped only vaguely. Neither is it coincidence that students cannot read their texts.

Lesson: One type of sheet of glass lets through 96% of the light that hits it. How much light goes through three sheets of that glass? Ten sheets? Give a mathematical model for the amount of light passing through any number of sheets of this glass.

Answering the question about three sheets requires some understanding and some computation. The original amount of light is not given, so the answer is only a percentage, not an actual amount. The model, "$A(n) = A(.96^n)$" [you may use different letters], uses a parameter, $A$, for the original amount of light. Try asking this problem on a quiz and you will see it is not easy for students. The idea of unspecified parameters is difficult, and even the use of some letter, here "$n$", to hold the place of "any number" is not trivial.

The purpose of the problem is only partly to illuminate the workings of exponential models. The greater purpose is to teach students how letters are used to express mathematical operations. It is a step toward teaching students to read.

Homework, quizzes, and exams can have a small fraction of questions which ask questions that emphasize reading, writing, and patterns.
Example 1: "State the *algebraic formulation of the method* used to simplify 3/(4/5) and 2/(5/13)." The answer is given by "$a/(b/c) = ac/b$" (if $b$ and $c$ are not zero).
Example 2: "Complete the theorem: If $x^2 + dx + k = (x + a)(x + b)$, then $d =$ ___ and $k =$ ___."
Example 3: "Give the equation of the line through the points $(a, b)$ and $(c, d)$." Because these letters are not the usual letters, the student must focus on the pattern.

Students who can do these types of problem have focused on the generalizations and patterns we want them to know. Such problems force the learning of mathematics.

Conclusion. Lessons can be used to discuss content and language simultaneously. Calculators save time which can then be used to focus on generalizing and remembering. Because symbolic language is critical to mathematical advancement, it is sometimes appropriate to have students summarize the essential point of a lesson by composing a theorem to express it. It helps them learn the result, and helps them learn to read and write mathematics.

## REFERENCES

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