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A General-Education Course Emphasizing Mathematical Language and Reasoning

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"We found that kids hit a wall, and that wall is called fourth grade. At that moment, a kid shifts from learning to read to having to read in order to learn."[1] Half of all children never make that transition.[2] These sorry thoughts are about reading English. Imagine how much worse the statistics would be if we determined the fraction of kids who can read *mathematics* to learn *mathematics.* This article is about an innovative approach to teaching the language of mathematics.

Recent calls for reform in college mathematics include revitalization of general-education courses designed for non-mathematics related majors (Mathematics Association of America, 1989, Sigma Xi, 1989, Steen, 1991). In spite of the recognition that "all well educated college people should be mathematically literate," (Mathematical Association of America, 1989, p. 109) most current undergraduate offerings to non-majors do little to effectively meet this goal. One school of thought suggests organizing courses around real and engaging problems which can serve to motivate students (Garfunkel, 1988). This article reports on The Language of Mathematics (Esty, 1991), a course at Montana State University which takes a different approach.

This one-term freshman-level course uses a distinctive instructional approach to study the language of mathematics in order to develop students' abilities to read with comprehension, to express mathematical thoughts clearly, to reason logically, to recognize and employ common patterns of mathematical thought, and to grasp the nature of proof (Esty, 1992). The course provides students the opportunity to both communicate about mathematics and to use mathematics to communicate, making explicit many of the recommendations in the NCTM *Standards* (1989) communication strand. A qualitative study found that students in this course were able to meet the course objectives. In addition, students improved their algebraic procedural skills and dramatically improved their attitudes towards mathematics.

The course discusses both linguistic and logical aspects of the language of mathematics. For example, the most important and profound linguistic feature of the language of mathematics is the concept of a placeholder — a dummy variable. Placeholders are fundamental to abstract symbolic mathematical expression.

One well-known use of "x" as a placeholder is in the definition of a function. When given the declarative definition "$f(x)=x(x+1)$," many students cannot express $f(\sqrt{x})$ or $f(x+h)$ because they do not grasp the concept of a placeholder.

Theorems use placeholders. For example, in the theorem, "$|x| < c$ if and only if $-c<x<c$," the "x" may represent "7" or "b" or "$5-2x$". Theorems are applied to cases where the letters in the theorem *represent*, but are not the same as, the letters or numbers in the problem. Without a clear grasp of placeholders, students cannot read theorems in textbooks. They are reduced to imitating their teachers, because they cannot grasp what the declarative sentences in theorems tell them to do.

The language of mathematics also has features from logic that are frequently misunderstood. For example, many students do not understand the essential connective ***if..., then...*** and how its use differs from the use of ***if and only if*** (equivalence). We, as teachers and authors, have not been clear about the distinction. Many texts state the additive property of equality using *if..., then...*: "If $a=b$, then $a+c=b+c$." This is true, but not precise. The teacher will, rightly, use the additive property as if it asserted equivalence. Later, when adding a constant is replaced by squaring (If $a=b$, then $a^2=b^2$), students see the same connective and expect the same equivalence. Imprecise usage of *if..., then...* undermines students' ability to understand our connectives and read our texts with comprehension. Students learn the unintended lesson that extraneous solutions can mysteriously appear. Few have any idea that the connective is the key to the potential appearance of extraneous solutions. Certainly their textbooks do not make a point of it, and most do not distinguish checking for mistakes from checking that is required because the connective was *if..., then...*.

These are only two of many examples of linguistic and logical difficulties students may have with the abstract language of mathematics (Esty, 1992). Until these and many other difficulties are explicitly addressed and overcome, students will not be able to read mathematics to learn mathematics.

In most courses very few homework or test problems address language difficulties. Traditional mathematics instruction has focused on product rather than process. Problem answers tend to be numbers resulting from the application of procedures rather than the procedures themselves. Many students can *apply* procedures, but relatively few can *express* procedures. For example, students may be able to solve a particular equation and yet be unable to generalize their thoughts to explain the solution *process*. The abstract symbolic language of mathematics is designed for such generalization. It permits processes to be efficiently described (if the reader is fluent). The Language of Mathematics course focuses students' attention on the *processes* of mathematics and how they may be expressed.

Nothing illustrates a course better than the questions which students are expected to be able to answer. Here are three questions from exams in the course which illustrate the emphasis on (a) how mathematics processes are expressed, (b) placeholders, and (c) the logical structure of mathematics.

Question 1. Justification of Processes: Consider the following solution to the inequality "$|7-2x| < 1$."

Solution: $|7-2x| < 1$ iff $-1 < 7-2x < 1$ [step 1]
iff $-8 < -2x < -6$ [step 2]
iff $4 > x > 3$ [step 3]

a) State the justification for each step.
b) Short essay: Is this a proof of something? If so, what? If not, why not?

Student 14's answer is given next. It contains two minor errors indicated by an asterisk (*). Brackets enclose comments to the reader of this article:

a) Step 1) $|x| < c$ iff $-c < x < c$.
Step 2) $a < b$ iff $a-c < b-c$
Step 3) $b < 0$ and $x < y$ then $bx > by$*
[A correct answer: If $b < 0$, then ($x < y$ iff $bx > by$)]
b) This seems to be a proof which would assert that $|7-2x| < 1$ is equivalent to the sentence $4 > x > 3$ or rather that the solution set to the problem $|7-2x| < 1$ is all x's > 4 and less than 3*.

Question 2. Symbolic notation: Solve for x (you may use the quadratic formula written on the chalkboard and rules from Chapter 4): $ax + 3x^2 = bc$.

[written on the chalkboard]: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Correct Response: $x = \dfrac{-a \pm \sqrt{a^2 - 4(3)(-bc)}}{2(3)}$.

Question 3. Logical Terminology: (Long essay) Discuss "equivalence" of sentences. Define it and distinguish logical equivalence from equivalence. Demonstrate that you know at least four types of reasons why sentences may be equivalent. What is the role of equivalence of sentences in mathematics?

The students' responses to these questions will be analyzed in the subsection on conceptual attainment.

The Course

The course is based on the premise that there are a limited number of modes of mathematical expression and thought which pervade all mathematics. These relate primarily to (a) the use of variables, (b) abstraction of methods into formulas, identities, and theorems, (c) the truth of sentences, and (d) patterns of mathematical reasoning. Although many articles in the mathematics education literature have addressed individual cognitive difficulties students have with these concepts, we are unaware of any other course accessible to general-education students which implements a comprehensive program to overcome all of them.

The course begins with a study of the definition of *abstraction* and discusses its importance in written mathematics. Basic material is then presented about numbers, sets, and functions — subject areas chosen to provide mathematical sentences and paragraphs for further discussion. In addition, these topics provide familiar contexts with which to begin making certain abstract concepts of the language of mathematics the explicit objects of study. This material includes initial forays into the uses of variables, logical connectives, truth, definitions, and replacement. How methods are expressed using formulas, identities, and theorems which express important patterns is made explicit, and practice with simple methods is initiated.

Next, truth-table logic is introduced and the basic logical equivalences that are most often used to provide alternative ways to express the same thought are examined. In contrast to truth-table logic as taught in some "finite" mathematics courses, however, the examples used are nearly all mathematical. This study of logic continues with applications dealing with the truth of sentences with variables and culminates with a discussion of the different reasons why two mathematical sentences may appear different yet express the same meaning.

The next topic focuses on logic for solving equations. A close connection with proof is established by emphasizing the role of connectives, the reasons why steps work, and how justifications are expressed. Simple algebraic equations provide a familiar context for this initial exposure to proof.

An explicit study of proof is presented at the end of the course only after students have had sufficient exposure to truth, connectives, and justification. At this time the roles of prior results, tautologies, reorganization, and definitions are thoroughly discussed.

Instructional Structure

Certain aspects of the instructional structure of the course facilitate the study of the language of mathematics. This structure includes the use of an algebraic context, an appropriate sequencing of prerequisite concepts, the connectedness of topics throughout the course, an emphasis on language and communication skills, and the use of examples and homework at an appropriate level of abstraction.

In the course the mathematical abstractions of language, logic, and proof are studied in the familiar context of algebra. Most of the examples that are used to illustrate new concepts are drawn from material that the students have all seen and employed (although perhaps not fully understood) in previous classes. There is explicit concentration on the *patterns* which must be abstracted from the essential similarities of the examples. The use of algebraic contexts makes it possible for even those students with weak backgrounds to study a new level of mathematics. In fact, by looking at familiar topics in a new, more abstract way these students are able to develop a new set of concepts at the same time that they deepen their understandings of previously encountered mathematical ideas.

A second significant feature of the course is the way in which prerequisite concepts are made explicit and appropriately sequenced to enable students to develop an understanding of the nature of proof by the end of the course. This is an important component of successful mathematics instruction, since "necessary lower-order concepts must be present before the next stage of abstraction is possible . . . [in order to] present to the learner a possible, and not an impossible task" (Skemp, 1987, p. 19-20).

A third significant feature of the course structure is that its topics form an interrelated whole. Initial concepts are repeatedly used to extend mathematical understanding of subsequent topics. For example, the logical connectives introduced early in the course take on a deeper meaning as they are applied later to mathematical sentences, and then become an important aspect of the theory of solving equations and proof. This interrelatedness enables students to continually improve and deepen their understanding of *all* the material as the course progresses.

Language plays a crucial role in concept development. "Language is useful because by the mention of a word parts of a structure can be called up" (van Hiele, 1986, p. 86). Therefore, a major instructional focus of the course is the development of appropriate vocabulary, semantic structure, and mathematical contexts for the language of mathematics.

Many students who are outstanding in other subjects cannot do mathematics because they do not understand it. These difficulties can be addressed, and cured, by asking students to consciously reflect on the linguistic and logical aspects of the language. The context must be the study of the language *per se* such that understanding, not a numerical answer, is the goal. Otherwise the pursuit of "the answer" replaces the quest for comprehension. Classroom dialogue and the use of extended-answer and essay questions on homework and tests provide students with examples of the level of conceptual learning expected of them and give them opportunities to develop facility in using appropriate mathematical language. Questions are designed to elicit information on complex, abstract ideas and mathematical relationships, requiring students to explain the mathematical meanings implied by procedures, to provide justifications for these procedures based on mathematical truth, and to understand precision in mathematical language.

A collection of problems from the course which illustrate the mathematical context and conceptual emphasis of each of the above topics is given next.

Sample Questions from Homework, Quizzes, and Exams

Bullets (•) indicate problems. Brackets enclose selected answers and comments to the readers of this article. Some answers are too obvious and others too long to reproduce here.

- Explain the difference between an equation and an identity. [An identity is an equation which is true for all values of the variable. In identities, equality is interpreted liberally, so that if both sides are undefined, they are regarded as equal. Thus $\sqrt{(4x)}=2\sqrt{x}$ is an identity, even though both sides are undefined for negative values of x.]
- What is the usual term for the truth set of an equation? [The solution.]
- Explain the difference between a sentence with a variable and a statement.
- How do we multiply fractions? a) Express your answer in English. [There are numerous possible answers in the imperative mode: "Multiply straight across," is perhaps the shortest, but it is neither precise nor clear to students who do not already know how.] b) Express your answer in mathematical notation. [In the declarative mode: $(a/b)(c/d)=(ac)/(bd)$. The point of this problem is to have the students learn to express methods by stating facts using placeholders, rather than always attempting to use English as in part (a).]
- Resolve the conjecture: $x > y \Rightarrow x^2 > y^2$. [False, $2 > -3$ but $2^2 < (-3)^2$.]
- To solve $|3x + 1| < 17$ what would you do *first*? Write your answer in the imperative mode. [Remove the absolute value signs and put the interior between -17 and 17.] b) Which theorem says you can do that? [$|x| < c$ iff $-c < x < c$] c) In part a), what do the letters "x" and "c" of the theorem represent? [x represents $3x + 1$, and c represents 17. The point is to have the students note how concise and precise the mathematical method of expression is in part (b) compared to the awkward English in part (a).]
- Express the following fact using different letters: $ab = 0$ iff $a = 0$ or $b = 0$.
- What notation would a mathematician use to define a function that takes any given number and adds three to it and then squares the sum?
- Suppose each of the following sentences is true. Which express *mathematical* facts, and which express facts which depend upon the particular things represented by the letters? a) $3(x + 5) = 3x + 15$, b) $3x = 12$, c) $|x| \geq 0$, d) $|x| < |x+1|$, e) (A and B) $\Rightarrow B$, f) (A or B) $\Rightarrow B$ [The context and appearance of variables is important to understanding. The capital letter "S" might indicate a set, whereas, in this text, the capital letters "A" and "B" represent statements in the context of truth tables. (a), (c), and (e) are mathematical facts.]
- Give the logical *form* of the sentence: If $|x| > 5$, then $x > 5$ or $x < -5$. [$A \Rightarrow$ (B or C).] Restate the assertion in a logically equivalent form. [(A and not B) $\Rightarrow C$: $|x| > 5$ and $x < 5 \Rightarrow x < -5$.]
- [Use DeMorgan's Law to] Give the negation of "$-5 < x < 8$." [$x \leq -5$ or $x \geq 8$. There is an implicit "and" in the original sentence.]
- Here is a sentence: "$|x - 5| > 2 \Rightarrow x > 7$." Give its *contrapositive*. Give its *negation*. Is it true? [Because it is an *implicit* generalization, its negation is an existence statement: "There exists x such that $|x - 5| > 2$ and $x \leq 7$." Thus the counterexample $x = 0$ (one of many) *proves* the negation true and the original statement false.]
- True or false? a) $bc > 25$ is equivalent to $b > 5$ or $c > 5$; b) $b < c$ is equivalent to $b + d < c + d$; c) $a = b$ is equivalent to $a^2 = b^2$. [F, T, F].
- Give the definition of "upper bound" [(using connectives): b is an upper

bound of s iff if $x \varepsilon S$, then $x \leq$ b.]

- Why does the section "On Definitions" concentrate on defining open *sentences* containing vocabulary words [e.g. "*b* is an upper ground of *S*"] rather than defining just the *words* themselves? [There are at least three reasons: 1) A term requires a context which is given by a sentence, 2) logic applies to sentences, not words, and 3) concept definitions are truth-based, not just image-based, and open sentences which serve as defining conditions can be true or false.]
- If one equation implies another, how do their solution sets compare? Can their solution sets be equal? If one has more solutions than the other, which is it? [The first might have fewer solutions, but all of its solutions will be found among the solutions to the second. They can be equal.] To solve the initial equation, would you prefer to have your sequence of equations connected by "$\Rightarrow$" or "iff"? Why? ["iff." It preserves the solutions exactly.]
- What is an "extraneous" solution? [When an initial equation is solved by using rules correctly to obtain a terminal equation, any solution to the terminal equation which does not solve the initial equation is called an extraneous solution. This may occur when the connection "$\Rightarrow$" is used.]

The instructions for solving (simple) equations in Chapter 4, "Logic for Solving Equations" are, "In the following homework the solution is not the only goal. Exhibit every step, exhibit the connective [*iff* for equivalence, sometimes "$\Rightarrow$" is necessary], and cite a rule." A typical problem ranges from a factorable quadratic to a harder problem where a square root must be eliminated: $x - 1 = \sqrt{x + 11}$.
[Extraneous solutions may arise when "$\Rightarrow$" is used, for example, when squaring.]

- What is the difference between "prove" and "deduce"? ["Deduce" means to employ logic properly to obtain a conclusion from given hypotheses. "Prove" means to employ logic properly to obtain a result *from results already accepted as true*. For example, from "Mars is made of green cheese" we can correctly deduce "Mars is a dairy food," but the former does not prove the latter.]
- Suppose "$A \Rightarrow B$" is true. Discuss whether A "proves" B. [No. Logically, "3+4 = 7 $\Rightarrow$ the Fundamental Theorem of Calculus" is true, but the hypothesis does not prove the conclusion. Also, "$A \Rightarrow B$" can be true without "B" being true, because it is an open sentence. For example, "For all x, $x > 5 \Rightarrow x^2 > 25$," but this does not prove "$x^2 > 25$" for all x.]
- Determine whether the steps are sufficient to deduce the conclusion. Steps: $H \Rightarrow A$. $H \Rightarrow B$. (A and B) $\Rightarrow C$. Conclusion: $H \Rightarrow C$. [They are.]
- a) What is the difference between theorem-proving and equation-solving? b) When you solve an equation, do you *prove* anything? If so, what? [Yes, you prove that the initial equation is equivalent to the terminal equation which exhibits the solutions.]
- What is a proof?

Conceptual Difficulties

In order to successfully read and write the language of mathematics and understand the nature of proof, it is necessary for students to develop a set of related concepts at an appropriate abstract level. Research has documented the existence of cognitive difficulties that can impede such learning (Balacheff, 1989, Bell, 1976, Dreyfus and Hadas, 1988, Martin and Harel, 1989, Tall, 1991). For example, even math-able students have profound difficulties with (a) mathematical language and notation, (b) logic (including quantifiers) and proof techniques, (c) concept definition versus concept image, and (d) perceptions of the nature of proof (Moore, 1991). In this section we present a brief discussion of these and other difficulties that are explicitly dealt with in the course to enable students to develop an understanding of the language of mathematics.

Perhaps the most important conceptual change required before students can become fluent in the language of mathematics and do proofs is the change from a procedural (*imperature* — Do this!) mode of thought to a truth-based, *declarative*, mode of thought. The declarative mode uses the fact that proofs are based on true steps justified by reasons, not on procedures. In the course the use of conjectures requires students to explicitly deal with truth and falsehood, as opposed to methods. This encourages them to take responsibility for the truth of their own steps whenever problems are to be solved (instead of having to rely on the instructor for judgement). Students must know and be able to state results (with variables) justifying their work.

Because the language of modern mathematics is designed to express methods (procedures) in an abstract declarative form (as theorems, identities, and "properties" using letters [variables] as placeholders), the use of placeholders and the correspondence between methods and statements must be explicit objects of study. Particular methods serve as examples, but the true object of study in the course is focused on how methods can be expressed.

The subject of logic is carefully examined in the course. Logic is, by definition, the study of connectives and quantifiers independent of the *meaning* of the component sentences. Thus statement *forms* (statement formulas) are the direct object of study, rather than statements. Students focus on the abstract study of the truth of compound sentences formed using logical connectives. Furthermore, instruction is designed to make explicit the fact that the truth of compound statements because of logical *form* is conceptually quite different from truth because of *meaning*.

The study of concept definitions is delayed until students have developed appropriate prerequisite understandings. Concept definitions employ mathematical language as well as connectives and quantifiers from logic in open sentences, so this area of difficulty cannot be resolved until language and logic have been addressed. A concept *image* is derived from examples, but is not formalized in the abstract manner that a concept *definition* is (Tall and Vinner, 1981). Therefore, in the course students must first develop an understanding of the essential components of a concept definition: connectives, logical form, open sentences, and equivalent sentences.

Both equation-solving and proofs require logically *connected* steps. Additionally, a key conceptual difficulty with proofs is that steps must be *justified* by reference to prior results. Once students have developed a thorough background in connectives and sentences, they are prepared to deal with paragraphs such as sequences of sentences used to solve equations. Logic for equation-solving employing the familiar context of algebra can be used to help students become accustomed to the practical use of logical connectives in paragraphs. By requiring students to justify algebraic steps by explicit reference to the basic results of algebra, students are given practice in this difficult requirement of proofs.

Students' perceptions of proof rely on a grasp of the two primary conceptual components of proof: (a) tautological *form*, and (b) prior results (component steps which are justified by results already accepted as true and which may include concept definitions). Proof techniques and the organization of proofs are addressed by theorems (tautologies and logical equivalences) from logic. Furthermore, proofs often address logically rearranged versions of the original theorem. One explicit object of study of the course is on how logical equivalences can be used to suggest the proper organization of proofs, including where to begin.

Without a grasp of all of the components discussed above — language, truth, the declarative mode, logic, concept definitions, and justification — the fundamental prerequisites for an understanding of proof are missing. The explicit study of proof must be deferred until students have acquired these necessary prerequisite concepts. Therefore, the formal study of proof is delayed in the course until the students have received a thorough grounding in the necessary preliminary concepts. This careful sequencing of topics then makes it possible to successfully teach this difficult area of mathematics.

Course Evaluation

Informal reports of the course's effectiveness and popularity in its first three years prompted a study of the course in its fourth year to determine the impact of the course on the students' conceptual attainment and their attitudes towards mathematics. Qualitative, as opposed to quantitative, techniques were selected to enable the study to be exploratory in nature, generating explanations for observed patterns of behavior rather than testing a set of pre-conceived hypotheses about classroom learning. Since "teacher-learner-subject matter interactions . . . must be at the very core of every educational endeavor," (Wheeler, 1989, p. 286), this methodology was used to investigate the complex triadic instruction-student-subject matter interactions that occurred.

The study was carried out over the eleven-week duration of the course in order to investigate the students' perceptions of themselves as mathematics learners and to examine their mathematical performance. Data consisted of observations of the class lectures, student interviews, students' written quizzes and tests, and a content analysis of the course material. The analysis of these data focused on identifying patterns of behavior and perceptions held by groups of students rather than on developing descriptions of the class as an aggregate whole, making it possible to study the variations in performance and attitudes that occur naturally in any classroom setting. The following sections present an overview of the analysis of the students' written work and a summary of that part of the analysis of the interviews that examined the students' affective responses to the course.

Even though The Language of Mathematics was designated as a freshman-level course, the 22 students in the study were predominantly upperclassmen (64% were seniors) and all but one were non-mathematics majors (Table 1). They were enrolled in the course in order to complete their minimum mathematics graduation requirement at Montana State University. Many of the students had weak mathematics backgrounds and had previously taken developmental mathematics courses at the university (Tables 2 and 3).

Table 1

Distribution of students by college class
2 - Freshmen or Sophomores
5 - Juniors
14 - Seniors
1 - grad stu. in math ed

Table 2

Distribution of students by high school math background
2 - general math
7 - through geometry
9 - through algebra II
4 - beyond algebra II

Table 3

Distribution of students by previous mathematics courses at Montana State University
7 - course equivalent to high school algebra I
5 - course equivalent to high school algebra II
4 - courses beyond algebra II level

Communication, Assessment, and Conceptual Attainment

This section presents a summary analysis of students' responses to selected test and quiz questions that were given within the final month of the term. The purpose of this information is to illustrate the levels of communication stressed in the course, the nature of the written assessment component, and the kinds of responses that were obtained from the students.

An important feature of the course was the way in which communication and assessment were used to promote instruction. Students were constantly required to speak and listen to mathematics through regular participation in class activities. The instructor frequently initiated dialogue by asking each student in turn to supply answers to questions from the text. This technique, used in a non-threatening way, provided practice in communicating mathematical concepts as well as encouraging the students to come prepared for class.

The instructor measured the students' progress informally through class questioning as well as formally through written quizzes and tests. However, students were informed that their overall course grade would be based primarily on their performance near the end of the term. This type of grading was used because the goal of the course was to develop long-term concepts and language facility (Esty and Teppo, 1992). Such concepts are abstracted from numerous examples which accumulate over the entire course, and it is expected that the students' language facility will steadily improve with practice.

From the beginning of the course, students were questioned on abstract concepts, logical relationships, and criteria for establishing if sentences are true. Their responses were judged in terms of mathematical correctness, precision of language (English as well as mathematics) and clarity of explanation. The instructor used the daily homework, class discussions, and early quizzes to not only monitor the students' conceptual development, but to establish standards of mathematical communication with the students. Feedback from this assessment provided the students with information on the nature of the knowledge that was to be learned in the course, and on the methods of communication that were to be used to express this knowledge.

The three questions discussed in this section illustrate the type of communication and level of concept attainment that was expected of the students by the end of the term. The responses presented indicate the students' abilities to express mathematical concepts clearly in English, use mathematical terminology appropriately, and use symbolic notation to express logical and mathematical relationships. The selected questions address the mathematical concepts of equivalence, the theory of solving equations, and the use of patterns and variables.

Concept Development

Essay questions were used to measure concept development. One example is question number 3 stated in the introduction. This question, given on the final exam, measures the students' abilities to distinguish between equivalence and logical equivalence (in terms of concepts, relationships, and uses), to use appropriate terminology, and to express their thoughts clearly in an extended essay.

The students' answers were analyzed on the basis of completeness of response, use of proper mathematical terminology, inclusion of four reasons (with examples) why things may be equivalent, and absence of errors. Nine students provided answers that were considered to be adequate to good, with scores ranging from 17 to 23 out of a possible 25 points. Twelve other students wrote less acceptable responses that included many errors or omissions, with scores ranging from 10 to 18.

The following two examples, written by Shane and Walter (pseudonyms), illustrate the range in answers that were written by students for this final exam question. Both are given verbatim to indicate the depth of response that was elicited by the question. Parentheses within the answer were part of the student's essay. Errors or vague assertions made by the students are indicated by an asterisk (*) and brackets enclose comments to the reader of this article.

-- Shane (student #22). Score, 23/25:
Equivalence is apparent when the truth sets of two sentences are equal. Equivalent sentences may be substituted for one another and the outcome will be the same. Equivalence deals with sentences and logical equivalence deals with sentences using logical connectives. Equivalence is used in the context of sentences in which a more understandable or usable form makes

progression of steps in the solving of a sentence more apparent or easier to visualize. Logical equivalence is used in the context of sentences which could be used in another form to make a progression of steps, as in a "proof," more easily understandable and also, sometimes, the "proof" can be made more concise and without unnecessary intermediate steps when logically equivalent forms are used properly. Four reasons things may be equivalent are: 1) theorems: A or (not A) is a tautology, $x \in S$ or $x \notin S$ is a tautology; 2) definitions: The negation of $A \Rightarrow B$ is $A \wedge$ (not B); the negation of $C \Rightarrow D$ is $C \wedge$ not D,* [This is not an example of a definition. "'$S \subset T$' is equivalent to 'All members of S are in T'" would be a definition of "subset."]; 3) logical equivalences: $E \Rightarrow F$ is logically equivalent to (not E) or F, and 4) quantified variables: $\exists P \ni P^2 - 10 = (P+100)/P^3$, $\exists c \ni c^2 - 10 = (c+100)/c^3$. Equivalence plays the important role of allowing substitution of equivalent sentences which will make finding solutions of sentences more concise and much easier to follow from subsequent steps. Logical equivalence does this as well when logic is employed as in the context of proofs and truth tables.

-- Walter (student #9). Score, 13/25:
Things that are equivalent often have the same truth values. Equivalence is important because it is a way to find solutions that otherwise might not have been possible. Things may be equivalent because of 1) Definitions, 2) logical equivalences, 3) Quantitative* [quantified] Variables, 4) Theorems. The role logical equivalences take in mathematics is that it lets us see the same thing in a different way and lets us find a solution by using different connectives and rules and theorems.

These two student answers illustrate the variation in mathematical content and terminology that students were able to produce in answering this question. As can be seen by Shane's lengthy response, the essay format provided an appropriate vehicle for assessing the degree of completeness of his development of a very complex concept. This question also demonstrated the students' abilities to communicate mathematical information in writing. Sixteen students (75%) used between three quarters and one full page of paper ($8^1/_2$ by 11 inches) to write their answer. Only five students (23%) used a half a page or less. Most students wrote in complete sentences in paragraph form rather than simply listing the required information.

Symbolic Notation

Question number 1 cited in the introduction (in which students were asked to express methods for solving inequalities) provides an example of the way in which assessment was focused on measuring students' understanding of process and methods of expressing processes rather than on measuring competence with symbolic manipulation. This question examines the students' understanding of the theory of solving equations, their knowledge of specific theorems related to the manipulation of inequalities, and their ability to express this information in appropriate mathematical notation. This question also tests their understanding of proof in application.

The students' answers were analyzed in terms of the numbers and kinds of errors made. Seven students made one or two minor errors with scores ranging from 16 to 19 out of a possible 20 points. Six students made two substantial errors with scores from 11 to 14 points. Seven students made three substantial errors with scores from 10 to 12 points. The remaining two students did not answer two of the four parts and received scores of 8 and 7.

Most of the students used algebraic symbols to state the theorem in each part of the question. A few gave the name of the theorem and described its properties in words. All students realized that some form of logical connective was employed in writing down the algebraic manipulations in each step of the solution process.

Question number 2 cited in the introduction (on the quadratic formula) examines the students' ability to recognize a mathematical pattern expressed in variable notation and to make appropriate substitutions within this pattern. Ten students answered the problem correctly. Seven students did not put the original equation in standard form and consequently failed to associate a negative sign with "bc." The remaining four students were not able to identify the appropriate values for substitution, assigning values to "a," "b," and "c" on the basis of the position of each term from left to right in the original equation.

This problem measured one aspect of the students' abilities to assign appropriate mathematical meaning to collections of algebraic symbols. This question represented a type of pattern recognition and substitution that many students found difficult to master early in the course. When a similar quadratic equation ($2cx+bx^2-5=0$) was given to the students at the end of the 7th week of classes, 56% of the students assigned values based on physical position of terms in the original equation. At the end of the term, however, only 19% of the students were still unable to assign values based on mathematical relationships instead of physical location.

The questions presented in the introduction and analyzed in this section illustrate the ways in which conceptual understanding was assessed in the course using quizzes and tests and the level of conceptual attainment that was expected in the course. Questions are designed to elicit information on complex, abstract ideas, mathematical relationships, and an understanding of the precision of mathematical language. Most questions are not procedural (that is, they do not just ask questions to *do* procedures), but are genuinely conceptual. Students are asked to explain the mathematical meanings implied by procedures and to provide justifications for these procedures based on mathematical truth. Within the eleven-week term, most students were able to develop an understanding of, if not necessarily a fluency with, abstract mathematical structure and reasoning. They were also able to communicate this understanding competently in English as well as in mathematics. In addition, many students improved their algebraic skills, because one result of studying the logic for solving equations was that it enabled the students to understand the mathematical reasons behind the equation-solving procedures that many had previously regarded as sequences of meaningless steps. This understanding enabled the students to solve equations with confidence, develop facility in factoring quadratic equations, and produced intense feelings of satisfaction.

Affective Responses

This section examines the impact of the course on the students' mathematical attitudes and beliefs through an inductive analysis of the transcriptions of the student interviews. An emergent constructivist framework within psychology regards individuals as meaning-seeking beings who create their own representational models of the world. These models in turn become the foundation upon which individuals assign meaning to new experiences (Mahoney and Lyddon, 1988). According to Noddings (1990), this framework implies that research must "investigate the subjects' perceptions, purposes, premises, and ways of working things out if we are going to understand their behavior" (p. 14-15).

Using this framework, the interview was selected as an appropriate method of data collection for studying the students' perceptions of themselves in a mathematical environment (Goetz and LaCompte, 1984). Many of the students were eager to share their feelings of anxiety, anger, and frustration over their past and present mathematics experiences once they discovered that the researcher was genuinely interested in listening to them. As the students' feelings changed throughout the course, the interviews provided the students with opportunities to continue to discuss their altering self-perceptions.

The students were individually interviewed approximately every three weeks during the course. All students were interviewed as least twice, some as many as four times. The students were informed that information from these interviews would not be shared with the course instructor until after the class was finished and that all their comments would remain anonymous. All interviews were tape recorded and transcribed for later analysis.

The particular questions used in each round were selected prior to the beginning of each set of interviews and were based on students' previous responses, instructional issues occurring at that time in the course, and suggested issues raised by the instructor. Interviews were semi-structured, but open-ended, using a set of initial questions for all students.

Following the completion of the course, the students' comments were analyzed by comparing and contrasting particular statements to identify patterns of responses that were shared by groups of students or that occurred as a function of time. Patterns were then characterized through descriptions of their common traits. The results of this analysis led to the identification of subjective dimensions of mathematics learning that represent one possible interpretation of the students' self-perceptions of their experiences and feelings in relation to the class.

The dimensions are labeled *attitude and mathematics involvement,* and *mathematical self-confidence*. These dimensions, which cover both affective and metacognitive behavior, do not describe discrete states but represent areas of mutual influence, with overlapping traits. They are organized as separate dimensions to provide differing perspectives on the complex interaction of affect and cognition within the mathematics learning environment.

Attitude and Mathematics Involvement

The dimension of *attitude and mathematics involvement* was used to characterize the students' affective interactions with mathematics and with the class activities. Seven categories were developed by comparing and contrasting the types of statements made by the students at the beginning and end of the course, differentiating in terms of the degree of positiveness or negativeness of each statement. In the most negative category students expressed high levels of anxiety and indicated a strong avoidance of mathematical participation. At the other extreme students expressed a marked enjoyment of mathematics and indicated a strong involvement in mathematical activity. The seven categories that were developed and their classification criteria are as follows:

1. *High Anxiety and Mathematics avoidance:* Students expressed debilitating emotional reactions to mathematics or the class situation.
2. *Moderate Anxiety and Mathematics Avoidance:* Students expressed moderately debilitating emotional reactions to mathematics and/or described avoidance behaviors.
3. *Dislike of Mathematics:* Students expressed dislike of mathematics but gave little indication of avoidance behavior.
4. *Indifference*: Students expressed mild dislike or neutral feelings about mathematics.
5. *Slight Enjoyment of Mathematics:* Students expressed mild enjoyment of mathematics.
6. *Moderate enjoyment and Mathematics Involvement:* Students indicated that they were comfortable with doing mathematics, had a sense of understanding, and experienced moderate enjoyment of mathematics.
7. *Enjoyment and Enthusiastic Mathematics Involvement:* Students expressed enjoyment and confidence in doing mathematics, a feeling of understanding, and strong positive reactions to the subject.

Table 4 indicates the change in attitude and involvement and final course grade for each student.

During the course the students, overall, exhibited a marked improvement in attitude towards and involvement in mathematics. At the beginning of the class 50% of the students had either high or moderate anxiety. With the exception of the mathematics education graduate student, at the beginning the most positive attitude toward mathematics was one of indifference (expressed by 23% of the students). In contrast, by the end of the course indifference represented the most negative attitude (held by 23% of the students). At the end, 38% were classified in the most positive category, and 64% of the students were classified in the two most positive categories.

On an individual basis all but five of the 22 students exhibited an improvement in their feelings towards the subject. Four of these five were classified as indifferent both at the beginning and at the end of the course (and the fifth was the mathematics education graduate student who was already in the top category at the beginning). All but two of the 11 students in the two most negative categories (#1, #2) at the beginning of the course moved up into the top two categories (#6, #7) by the end of the class.

Table 4
Individuals student's final course grade and changes in attitude and involvement

Student	Grade	1	2	3	4	5	6	7
				Category of Response				
1	C		O				X	
2	B				■			
3	C	O					X	
4	B		O					X
5	B				O	X		
6	B				■			
7	A							■
8	A			O				X
9	D				■			
10	C		O				X	
11	C			O				X
12	A	O					X	
13	A	O						X
14	C		O				X	
15	C	O						X
16	A				■			
17	C		O					X
18	A		O			X		
19	C			O		X		
20	C		O		X			
21	C			O				X
22	A			O			X	

O = Initial category at beginning of class
X = Final category at end of class

1 = High math or performance anxiety and math avoidance
2 = Moderate mathematics anxiety and avoidance
3 = Dislike of math
4 = Indifference
5 = Slight enjoyment of mathematics
6 = Moderate enjoyment and math involvement
7 = High enjoyment and math involvement

The following comments illustrate the types of changes that students experienced by the end of the course.

> *Beginning*: "I wouldn't be here if I didn't have to. I'm frustrated, angry, hating it. I get real tense all the time." (Alice - category #1.)
> *End*: "I'm doing way, way better than I thought I ever would. I think if I ever did take math again I could do better than I did before. I think I can probably understand it better." (Alice - category #6.)

Beginning: "I usually dread studying math. When I took another math class I forced myself to sit down and study. I hated every minute of it." (Andy - category #3.)
End: "I'm enjoying this math class more than I've ever enjoyed a math class before. I'm studying. I'm not dreading sitting down to do my homework. I'm doing it every day and I look forward to doing it." (Andy - category #7.)

Mathematical Self-Confidence

The dimension of *mathematical self-confidence* describes an individual student's perceptions of his or her ability to successfully perform mathematical tasks. This dimension combines a metacognitive element related to students' perceptions of learning with an affective trait of self-confidence. The characteristics of this dimension were developed from the interview transcripts by classifying statements as either "confident" or "non-confident" and subdividing these two categories along common traits.

The traits do not represent an ordering of perceptions from negative to positive. Instead, they should be regarded as parallel characteristics describing different aspects of the students' perceptions. Many of these characteristics share attributes classified under the dimension "mathematics involvement." Descriptions of these traits are given below. The number of different students who made such statements at some point in the course are given in parentheses.

-- Non-Confident

1. *Inaccessibility of Mathematics* (8): Students indicated that forces beyond their control impeded their ability to do mathematics. They also perceived mathematical activities and learning to be meaningless.
2. *Lack of Accomplishment* (10): Students indicated that they perceived few positive results despite working hard. They also indicated they lacked specific skills and knowledge.

--Confident

3. *Accomplishment* (15): Students indicated that they could perform specific mathematical activities and commented that, by working hard, they were able to succeed.
4. *Confidence* (14): Students expressed feelings of confidence over their performance in the class and commented on how motivated they were to do mathematics.
5. *Understanding* (16): Students commented on their ability to understand the course material.

Many students' perceptions of their mathematical confidence changed from the beginning to the end of the course. During the early part of the course 14 students (64%) characterized themselves as mathematically non-competent to some degree based on their experiences in previous mathematics classes or on their early performances in the class. By the final interview, however, all students had at some point in the course used statements of self-confidence to describe their present behavior. Only one student was still referring to himself in non-confident terms by the final interview (Table 5).

Note that the interview process was designed to elicit the concerns and beliefs of individual students, not to prompt them to confirm or deny the interviewer's preconceptions. Not all students expressed the same concerns when responding to the initial, pre-selected questions, so the follow-up questions did not always touch on all the concerns later identified as significant. Therefore, blanks occasionally occur in the descriptive tables of student responses (see Table 5).

Table 5
Students' perceived mathematical competencies as a function of time

Student	Previous Math	Interviews I	II	III	IV
1	0		X	X	X
2		-	X		X
3		-	0		X
4	0		X	X	X
5		-	X		X
6		-	X		X
7		X			
8	0	-	X	X	X
9		-		-	X
10	0		0	X	X
11			0	0	0
12		0	0	0	X
13		0	X	[illegible]	[illegible]
14		0	-	-	X
15	0	-	0	X	X
16			X		
17	0	0	X	X	X
18	0		X	X	X
19	0	-	X		X
20		-	X	-	X
21		X			X
22	0	-	X		X

- = Not interviewed
0 = Non-competent
X = Competent
blank = Not discussed

Eight of the 22 students (36%) did not show a marked change in their self-perceptions throughout the course, making only confident statements when interviewed. These eight students represented all those classified at the end of the course as indifferent or expressing only slight enjoyment of mathematics. These students, who already felt comfortable with their mathematical performances at the beginning of the class, did not change much during the course along the subjective dimensions of learning that were analyzed. It is argued that the unanticipated success in the course of the 13 students with initially low levels of mathematical confidence contributed to those students' marked improvements in attitudes towards and involvement in mathematics (Teppo, 1992).

The following comments illustrate the nature of two students' mathematical self-confidence and the way that this changed from the beginning to the end of the class.

Doug (#15):

- Second interview - "To do math - there's got to be something up there. There's something innate in there. For five years I've been thinking - how am I going to be able to do math?" (Non-confident.)
- Fourth interview - "Working this hard, I've proved to myself that I can do whatever I want, I can do anything. I've learned from this class that it's not what you cannot do, it's what you can do." (Confident.)

Doug's statements illustrate a dramatic shift in locus of control. At the beginning of the course, he indicated that mathematical ability was innate and, by inference, beyond him. At the end of the course, he connected his success in the class to hard work. During the final interview, Doug expressed great confidence in himself and motivation to perform well in the few remaining weeks of the class. These perceptions of his mathematical competence parallel his beginning and end classifications on the dimension of attitude and involvement (#1 to #7).

Mary (#6):

- Second interview - "The text is clear enough that the homework problems are clear." (Confident.)
- Fourth interview - "I haven't been challenged with anything different. I don't find the class difficult." (Confident.)

Mary was classified as indifferent in her attitudes and involvement both at the beginning and the end of the class. The fact that Mary's performance matched her expectations produced far less affective impact on her than did Doug's reversal of classroom expectations.

Discussion

Research dealing with attitudes, beliefs, and emotions in mathematics education has recently been focused on examining these factors in problem-solving situations either within clinical interview settings or within small group classroom interactions (McLeod and Adams, 1989). These studies seek to understand the roles that affective variables play in students' abilities to solve mathematical problems. This study takes another perspective, examining attitudes and beliefs within the context of actual classroom mathematics instruction to develop characterizations of the types of attitudes and beliefs that occur *because of* mathematics learning.

The identification and characterization of the subjective dimensions of learning illustrate the complex, interactive nature of students' perceptions of themselves as mathematics learners, their beliefs about mathematics, and their affective reactions to the subject. The mutual influence of these variables indicates the importance of regarding mathematics learning as a set of complex, interrelated cognitive and affective factors (Lester, Garofalo, and Kroll, 1989).

This study found that students' increased confidence in their abilities to do mathematics was linked to an increased enjoyment and active participation in mathematical activities. At the same time, students' increases in understanding and valuation of the subject were related to increased confidence. These positive attitudes were reflected in decreases in the levels of students' mathematics anxiety.

These findings indicate that the course exerted a positive affective impact on many of the students. Dramatic shifts occurred within a limited instructional time in their attitudes towards and involvement in mathematics, their levels of self-confidence, and their beliefs about mathematics. A comparison of the changes in attitudes and beliefs of the "indifferent" students with those actively involved in the course by the end of the class indicates that the unanticipated success in mathematics experienced by many of the students may have been a powerful factor in altering their attitudes towards and involvement in mathematics.

Undoubtedly, the classroom atmosphere, instructor's dynamic teaching style, and the use of a progressive improvement grading scheme played a major role in reducing the students' initial levels of mathematics anxiety. What is interesting is that these students were also actively engaged in the particular subject matter. Students responded positively because they were able to understand the course material. It was not necessary for the content to be directly applicable to their daily lives or to consist of realistic and "fun" application problems. The mathematics itself became a motivating factor for class participation and learning.

The findings reported in this section substantiate the importance of considering the affective aspects of classroom mathematics instruction. Motivation to persevere and an awakened enthusiasm for the subject are student traits that have profound impact on increasing enrollment in mathematics programs. One of the goals of "Challenges for College Mathematics" (MAA and AAC, 1990, pp. 15-16) is stated specifically in affective terms: "Building students' well-founded self-confidence should be a major priority for all collegiate mathematics instruction."

Conclusion

Traditionally, mathematics instruction has been focused on product rather than process. These products, however, are only the final step in a long chain of thinking and reasoning; a chain of understanding and concept development that is omitted from much mathematics instruction. Students "have been taught the products of the activity of scores of mathematicians in their final form, but they have not gained insight into the processes that have led mathematicians to create these products" (Dreyfus, 1991, p. 28). Students are given a language, symbol system, and mathematical structure, but not the keys to unlock their meanings. "A language must be created: symbols defined, rules of manipulation specified, the scope of mathematical operations delineated" (Hanna, 1991, p. 60). The Language of Mathematics course addresses these issues.

The findings about the course based on the analyses of the students' written work and interviews are significant because they demonstrate that it is possible to develop a course that systematically teaches abstract concepts of mathematical expression and reasoning. Furthermore, this type of instruction is readily accessible to so-called "weak" students and has been shown to have a positive affective impact on them. The Language of Mathematics course is unique in that it *comprehensively* addresses areas of cognitive difficulty related to the linguistic and logical structure of mathematics. We argue that the focus and sequencing of the course content and instructional emphasis, the levels of mathematical understanding developed by the students, and the attitudinal changes they experienced are all interrelated factors contributing to the success of the course.

An important factor is that the course emphasizes understanding, not just "doing." Mathematics is demystified as the students begin to understand its logical structure and observe the ways in which this structure operates in the theory of solving equations. A source of satisfaction for many of the students in the study was the fact that, under this type of carefully structured learning, they were able to truly understand mathematics for the first time.

The course content was originally designed to address the cognitive difficulties "good" college-level students have with mathematical reasoning and proof, but, apparently, these difficulties are largely the same difficulties that "weak" students have. Of course, the so-called "weak" students have additional difficulties at a less abstract level, the procedural level, but this does not appear to be a disabling handicap when studying this material. When approached as in The Language of Mathematics, these topics, which traditionally have been offered only to math-able students, also comprise an effective course for math-anxious, math-avoidant individuals who are traditionally neglected in the service population of collegiate mathematics instruction.

We conclude that mathematical logic and fundamental concepts of mathematical expression and structure need not be considered as esoteric topics restricted to upper-level college classes. It is possible to develop a rigorous course that attends to the linguistic and logical aspects of mathematics that is accessible to a wide range of students. The course content is fundamental to mathematics and applicable to all mathematics courses.

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NOTES

[1] Children's Television Workshop President David Britt, quoted by Miriam Horn in "Can the Boob Tube Finally Get Serious?" in U.S. News and World Report, August 24, 1992, page 61.

[2] According to a Harvard professor of human development, Colette Daiute, cited in the article in note 1.