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# WHAT DO WE NEED TO TEACH ABOUT ALGEBRA, NOW THAT "CALCULATORS CAN DO IT ALL"?

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What do we need to teach about algebra, now that "calculators can do it all"? When is it all right to use calculators? It is obvious that at some stage using sophisticated tools is all right. No one will object if a math professor uses a calculator to multiply and divide. He has paid his dues. The effort to do arithmetic by hand would be a waste of time. On the other hand, many would object to a second-grader doing multiplication homework with a calculator. The child's practice is regarded as productive — even essential.

Doing more problems has always been regarded as good for learning, and calculators certainly make it possible to "do" far more problems in the same amount of time. Isn't that good?

Maybe not. Doing 50 multiplication problems with a calculator is not the same as learning about multiplication. Exercises done with calculators might rapidly explore numerous examples designed to lead to profound generalizations about multiplication. But not if the goal is simply to obtain 50 individual numerical answers. Fifty calculator problems might produce almost no experience with multiplication unless there is some reflection on any patterns in what has happened, and why. Mathematics depends upon pattern recognition. This is related to Polya's often-neglected problem-solving advice: look back. If there is no analysis or attempt to find a pattern worthy of generalization, individual problems remain individual and do not help students develop and internalize the greater pattern we identify as learning.

Learning requires experience. Generally, in any subject, students who study twice as long gain more experience and do better. Of course, there are numerous major caveats. But teaching would be more effective if there were some way to concentrate more memorable experience on essential points in the same amount of time (Swartz 1996).

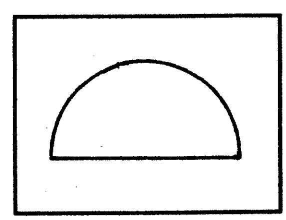
Clearly calculators can dramatically increase the rate at which students can gather experience. In algebra the focus is certainly not arithmetic. Calculators can keep arithmetic from stealing time from the point of an algebra problem. Even if the students do not have calculators, the teacher with a calculator and an overhead projector can sometimes provide memorable experiences to an entire class. It is possible to show someone how and why some procedure works and make the point with examples that can be more numerous because of calculators. But lessons must have some pattern or conceptual focus toward which the students are being led. Effective lessons must address essential patterns of algebraic thought. Only then can technology be effectively brought to bear on the problem. Otherwise we risk giving students the algebraic parallel to the ability to "do" 50 multiplication problems without learning any lesson about multiplication.

But, if someone grasps the patterns of algebra and understands its processes, we can hardly object to the time-saving aspect of using a calculator. However, many would argue that for students at a certain stage, using a calculator would short-circuit the learning process. We need to determine what the professor knows that beginning students do not know. We will see that this is not just a number of skills and rote procedures. Even a computer-algebra-system has to categorize problems correctly before it can select and do the appropriate procedure. Again, patterns describe the categories. It turns out that certain concepts are essential to pattern recognition and their development is closely associated with the symbolic language mathematicians use to describe patterns. These concepts, to be outlined below, are essential to the classification of algebra problems. Whenever calculators can help develop these concepts and patterns, they are valuable teaching tools.

What is algebra? What are its essential concepts? Concepts are mental objects. In mathematics, they are things with abstract reality. You can touch five blocks, but you cannot touch the number *five*. The *five* in "five blocks" is an adjective. In arithmetic the number *five* is abstract and divorced from physical context. It is a new type of object, a noun. Numbers (as nouns) are the essential concepts of arithmetic. The processes of arithmetic involve operations such as multiplication, but these operations take on conceptual reality only when their properties are recognized. "$4 \times 5 = 20$" is a fact about numbers. "$4 \times 5 = 5 \times 4$" is an instance of an algebraic fact about multiplication, "$ab = ba$." It is easy to imagine a set of exercises, facilitated by calculators, designed to elicit that fact, and thereby begin to provide the operation of multiplication with properties, and therefore begin to give the operation conceptual reality. The new conceptual objects of algebra are mental objects, operations and order, at this higher level of abstraction.

Even the simplest algebraic problems rely on the conceptions of operations and order. To solve "$3x + 5 = 17$," the key is to note the expression "$3x + 5$" and recognize that it expresses "Multiply by 3 and then add 5," operations which can be undone by "Subtract 5 and then divide by 3." If the equation had been "$3x + 5 = 92.4$," the solution *process* (but not the solution) would be the same. The process does not depend upon the the numerical value of x, or the numbers 17 or 92.4, which are irrelevant. Rather it depends upon the operations and order in the expression. Mathematicians call this type of conceptual object a function.

Research shows that there is a huge difference between being able to select and evaluate operations when the focus of attention is numbers, and being able to abstract and represent the operations themselves. The ability to express operations is essential to algebraic word problems (Esty and Teppo 1996; Teppo and Esty 1995).



Problem 1: A freestanding dog pen is in the shape of a semicircle (see the figure). If the diameter of the semicircle is 10 feet, what is the perimeter of the pen, including the diameter?

Precalculus students do not know a formula for this perimeter, but most can compute the perimeter anyway. The words in the problem and a basic formula suggest the right operations to do. The semicircle contributes half the circumference of a circle of diameter 10. So it contributes $10\pi/2$. The diameter contributes 10. The perimeter is $10\pi/2 + 10$. This seems like arithmetic. It does not require algebra. It is a direct calculation that requires the use of a formula, but does not need any unknown "x".

The same operations will work for any diameter, which is what an algebra student needs to recognize to do the following similar algebraic problem:

Problem 2: A freestanding dog pen is in the shape of a semicircle (see the same figure). If the perimeter of the pen, including the diameter, is 40 feet, what is the diameter of the semicircle?

The words in the two problems are almost the same, and the mathematical relationship is exactly the same. Nevertheless, this problem is much harder than Problem 1. Many students who have taken three or fours years of high school math can not do it. Why not?

The basic formula $D = \pi d$ again suggests *Multiply by π*, but that operation can not be executed, since the diameter is not known. *Semi*circular suggests *Divide by 2*, but we can not do it; the number is not given. To evaluate the perimeter, *Add*, but again we can not do it. Now we see why this problem is algebra and Problem 1 is not. In Problem 1 we just do the operations; there is no need to represent them. There is no need for the operations themselves to take on conceptual reality. The work focuses on the numbers obtained, not on operations. A problem is said to be <u>direct</u> when the words, symbols, or basic formulas express the operations you actually do to solve the problem (Esty 1997). Problem 1 is direct. We simply execute the suggested operations. The steps are arithmetic and there is no need to create a symbolic expression for the perimeter. However, in Problem 2, we do not do the suggested operations. A problem is <u>indirect</u> when the words, symbols, or basic formulas suggest operations you are not supposed to actually do. Instead, you represent them in symbolic notation and then manipulate (reorder) the operations. We must identify the operations and order (which requires conceptualizing them) and use the language of mathematics to express them. The student must follow the advice, "Build your own formula" (Esty 1997). Translating the suggested operations into symbols:

(1) $$P = \pi d/2 + d.$$

To find the diameter we must set the perimeter equal to 50,

(2) $$\pi d/2 + d = 50,$$

and solve for d. The first step depends entirely on the operations in the expression "$\pi d/2 + d$" and has nothing to do with the particular number 50.

(3) $$d(\pi/2 + 1) = 50.$$

(4) $$d = 50/(\pi/2 + 1).$$

In every step the focus is operations, not numbers. The steps would be the same if the original number had been different. The solution *process* is not dependent upon the number to which it is applied. Furthermore, the process does not use the operations in the formula. On the contrary, those operations first must be exchanged for other operations in a different order, so (2) is replaced by (3) and then the inverse of the last operation is performed to replace (3) by (4). This is algebra because the solution process concerns operations and order. This is algebra because operations are represented, but not executed. This is algebra precisely because the problem is indirect. This is algebra because the essential concepts — the mental objects — are operations and order.

Now we can see why algebra and word problems are hard and why the language of algebra is more than just symbols used to write concepts from ordinary language. The focus is not numbers. Although algebraic symbolism may have the appearance of representing numbers, its essential concepts — operations and order — are not concepts from ordinary language. Using the new language requires substantial conceptual development, not just translating from ordinary language. Building your own formula requires your attention to be concentrated on operations and order, which are new (and difficult) concepts. Expressing the concepts requires the language. Language and concept development go together.

The current curriculum pays scant attention to symbolic language. For example, students are rarely required to write procedures in the language designed to write them. (And, when they are, the context is usually word problems. which is not the easiest context in which to begin.) Identities describe some procedures. How do you add fractions? "$a/b + c/d = (ad + bc)/(bd)$." How do you subtract a larger positive number from a smaller positive number? "$a - b = -(b - a)$." More practice with patterns of operations would force and facilitate development of operations and order into concepts — concepts that the professor has that students do not have. We could ask students to state, in abstract symbolic notation, the theorem that expresses the method for the *first* step in solving "$\log(x^2) - 1.2 = 2.3$." One answer could be, "'$x - a = b$' is equivalent to '$x = b + a$.'" Yes, the "x" in the theorem can hold the place of "$\log(x^2)$" in the problem. The problem-*pattern* in the theorem ($x - a = b$) focuses attention on the essential operation for the first step. Examples like these force pattern recognition and concept development, which are facilitated by study and use of the language in which dummy variables (placeholders), unlike unknowns, are used to express thoughts about operations, not numbers. The well-known difficulty students have reading the definition "Let $f(x) = x^2$," and then expressing $f(x+h)$ shows that operations and order are not natural, or trivial, concepts. A sequence of lessons on algebraic language and concepts has been described elsewhere (Esty 1996, Esty 1997).

With the above distinction between algebraic thought and arithmetic, algebra lessons can be designed to focus attention on the essential points that distinguish professors from students. The symbolic language deserves more attention. And graphing calculators can be used to gain more experience with algebraic thought in the same amount of time. Articles too numerous to mention present interesting graphing-calculator lessons. Lessons should be evaluated according to whether they help develop algebraic mental categories — whether they facilitate abstraction and generalization — or whether they are simply cute individual activities.

As one illustration, consider the mental category "quadratic equation." This category is often too narrowly interpreted by students because they have not really conceptualized operations apart from the particular symbols used to express them. For example, many students who are perfectly capable of using the quadratic formula to solve "$x^2 + 3x = 11$" are incapable of using it to solve for y in "$x^2 + 2xy + y^2 = 20$," which is necessary for that equation to be graphed in the "y = ..." form required by graphing calculators (Teppo and Esty, 1996). Pedagogical problems that require and promote algebraic thinking like this one would be unusable without graphing calculators. On the other hand, they would be pointless with a computer algebra system that can do it by itself.

Therefore, when algebra is the subject, graphing calculators should be used to promote conceptual development at this new, higher, level. However, using their equation-solving feature must be ruled out when the point is to manipulate operations algebraically. Other than that, graphing calculators do nothing to short-circuit the development of algebraic concepts. On the contrary, they help fit far more algebra into the same amount of time. Surely lessons can be designed so that even symbolic manipulators will be able to contribute to the development of essential algebraic concepts. But the parallel with using calculators to do 50 multiplication problems is disturbing. For the student to know what the professor knows, algebraic mental objects must have personal reality to the student. This probably occurs only though the (well-guided) individual effort of mental categorization of numerous examples.

Doing algebra requires procedures that we all know can be close to mindless. Students almost always can "do" algebra long before they truly understand the underlying concepts and the abstract symbol system of algebra. Using a language well requires grasp of its nouns — it objects — which in the case of algebra are higher-level abstractions than numbers. The language and its mental objects are what we can hope students learn in algebra. This symbol system and its objects will remain, even after calculators can do it all.

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