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Algebraic Thinking, Language, and Word Problems

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MANY students progress through four years of high school mathematics and yet emerge at the end without acquiring the ability to solve word problems (Kieran 1992; Simon and Stimpson 1988). This is not simply a missing skill but an indication of a fundamental deficiency in students' abilities to think algebraically. This article examines student performance on word problems as a way to illustrate the critical role that having an appropriate language plays in the student's ability to think and communicate mathematically.

An important aspect of communication is having some way to represent the object or concept under consideration. It is difficult to think about, let alone talk about, entities that cannot be represented by some kind of word, symbol, or picture. "Language is important because by mention of a word parts of a structure can be called up" (van Hiele 1986, p. 86). In mathematics, symbolic language fills a dual role as an instrument of communication and as an instrument of thought by making it possible to represent mathematical concepts, structures, and relationships (Kaput 1989; Esty and Teppo 1994).

ALGEBRAIC AND ARITHMETIC THINKING

Algebraic language is a concise and efficient medium with which to express mathematical thoughts. Any meaningful sentence using algebraic symbols represents a communication about some mathematical object. This article distinguishes two hierarchical levels of thought (arithmetic and algebraic) that affect students' abilities to think about these objects and use algebraic symbols.

Many students, if given an equation, can manipulate the algebraic symbols correctly. However, these same students are unable to set up such an equation given its relationships expressed in the form of a word problem (Kieran 1992). This deficiency is in part related to students' inability to move from arithmetic to algebraic thinking. A conceptual change needs to occur in students' thinking as they move from arithmetic to algebra. The focus of thought must shift from *number* to *operations* on numbers and *relationships* between numbers.

The following problems demonstrate the need to make operations, rather than numbers, the primary objects of consideration in finding solutions to certain types of word problems. The problems also illustrate the distinction between algebraic and arithmetic thinking, and how the use of algebraic language facilitates thinking at a more abstract level.

Problem 1: A circular walk is inscribed in a square block. The area outside the circle and inside the square will be a garden (fig. 7.1). If the side of the square is 100 feet, how many square feet will the garden be?

Problem 2: A circular walk is inscribed in a square block. The area outside the circle and inside the square will be a garden (fig. 7.1). If the area of the garden is 1 400 square feet, how long is the side of the square?

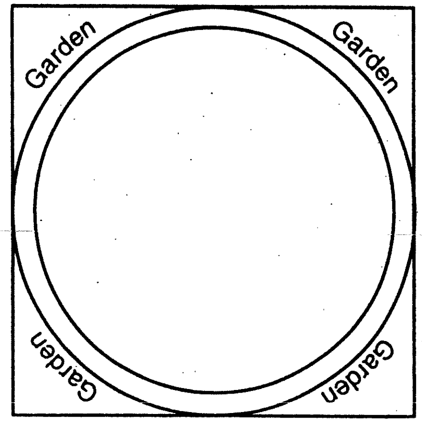

Fig. 7.1. Quiz figure for problems 1 and 2

Both problems describe the same quantitative relationship in words and ask for a numerical answer. Problem 1 asks for an area when a side is given. Problem 2 asks for a side when an area is given.

Each step in problem 1 is a numerical calculation using a well-known formula: 100^2 is the area of the square; 100/2 is the radius of the circle; $\pi(50^2)$ is the area of the circle; $100^2 - \pi(50^2)$ is the answer. At no stage do we need to use a symbol, such as x, to represent an unknown.

The operations are apparent in problem 1, but they are not the explicit focus of attention. The sequence of operations is the following: (1) *square* the side of the square; (2) separately *divide by 2* to find the radius; (3) *square* the radius; (4) *multiply by* π to find the area of the circle; (5) *subtract* the area of the circle from the area of the square. In problem 1, attention is focused on the numbers—the results of the calculations carried out at each step.

Problem 2 represents a different level of thought. To find the side when the area is given, the student must *represent* this sequence of operations symbolically without actually executing any calculations. Algebraic notation is designed for this job and makes it possible to "build a formula" for the area of *any* garden of that shape (regardless of the side of the square):

(1) $$A(x) = x^2 - \pi(x/2)^2$$

Then, according to the word problem,

(2) $$x^2 - \pi(x/2)^2 = 1400.$$

Notice that the remaining steps in the solution manipulate operations (but do not perform calculations):

$$\begin{aligned} x^2 - \pi(x/2)^2 &= 1400 \\ x^2 - \pi(x^2/4) &= 1400 \\ x^2 - (\pi/4)\,x^2 &= 1400 \\ (1 - \pi/4)\,x^2 &= 1400 \\ x^2 &= \frac{1400}{1-\frac{\pi}{4}} \\ x &= \sqrt{\frac{1400}{1-\frac{\pi}{4}}} \end{aligned}$$

The solution *process* (but not the solution) would be identical if the area were some other number besides 1400. Numbers are not the focus of attention. This problem illustrates the purpose of algebraic notation, which is to represent operations and order (without actually doing those operations). The critical steps are to represent the relevant operations symbolically and then to manipulate the operations. (The only alternative to using algebraic symbolism is to repeatedly use the operations with specific numbers in a guess-and-check procedure.)

Why are word problems thought to be hard? Is it because the relevant mathematical relationships are difficult to extract from the words? Problems 1 and 2 can be used to investigate that possibility because the relevant mathematical relationships are identical and even expressed in identical words.

Simply using words to describe a problem does not necessarily make it algebraic. If students can do problem 1 but not set up the algebraic equation (2) in problem 2, they are not thinking algebraically. Algebraic thought would be reflected by the use of symbols to help represent the essential conceptual objects (operations and order) of problem 2.

A Study

At Montana State University the students enrolled in our precalculus class have all taken three or four years of high school mathematics and intend to take engineering calculus, but their placement scores are too low to allow them to enroll in calculus. That is, these are students who have taken enough algebra for calculus but who, nevertheless, do not perform well at the prerequisite level.

Experience shows that almost all students who enroll in precalculus perform well at problems that require only plugging in to well-known formulas, regardless of the number of steps, but most are unable to do even short algebraic word problems if the relevant formula is not well known. We began to hypothesize that the difficulty these students had with word problems was not in the understanding of the mathematical relationships expressed in the words, but with the underlying algebraic concepts and the use of symbolism required to express these concepts.

A short action-research study was carried out to investigate this hypothesis. On the first day of one term of precalculus, all 137 students in five sections, mostly freshmen, were given either quiz 1 or quiz 2 (below), each containing one arithmetic and one algebraic word problem. One problem was to evaluate the area of a figure given a side, and the other was to solve for a side given an area. The pictures and the layout of the two quizzes were identical. The difference was that the types of problems were switched on the two quizzes. Problems 1 and 4 use arithmetic thought. Even though each requires several steps, each operation is simply a numerical calculation. In contrast, problems 2 and 3 use algebraic thought because the work requires operations to be represented without actually executing them.

Quiz 1

(Problem 1, discussed above.) (Find the area given the side.)

Problem 3: A pen has the pictured shape (fig. 7.2). If the enclosed area is 32 square feet, what is the length of the top side in the picture?

Quiz 2

(Problem 2, discussed above.) (Find the side given the area.)

Problem 4: A pen has the pictured shape (fig. 7.2). If the length of the top side in the picture is 3.5 feet, what is the area of the pen?

The students' responses to these two quizzes were evaluated and categorized according to whether operations were employed appropriately. The students' use, or lack of use, of a variable to express operations was noted in the solutions to the algebraic problems (2 and 3).

The responses to all four questions were categorized according to whether they exhibited the following characteristics: correct operations in the correct order (category A); some correct operations, but not all correct (category B); no correct operations (category C).

For example, in problem 2, those students who wrote an incorrect or incomplete equation or expression that nevertheless contained x^2 were placed in category B because "x^2" represents the necessary operation of squaring. The mere writing of the formula $A = \pi r^2$ in problem 2 or $A = (1/2)bh$ in problem 3 did not classify a response in category A or B if the formula was not also tied to the given problem situation.

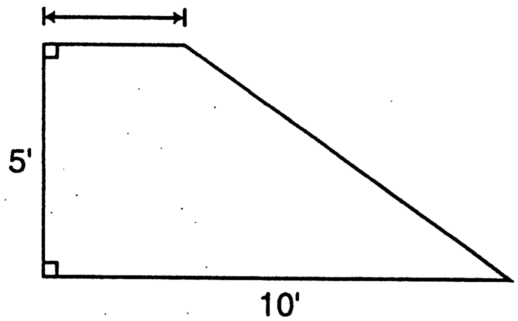

Fig. 7.2. Quiz figure for problems 3 and 4.

In problem 4 (an arithmetic problem), students who found only the area of the left rectangle (obtained by dropping a perpendicular to the base) were placed in category B because some operations were correct. In problem 3, the corresponding algebraic problem, students who set up a correct equation fell in category A. Those who expressed x and $10 - x$ or x and $5x$ were classified in category B if the rest of their answer was incorrect, because at least one correct operation was expressed. The distribution of student responses across all three categories for the four problems is shown in table 7.1.

TABLE 7.1
Numbers and Percentages of Students Using or Representing Operations Correctly: Results of 137 Freshmen Taking Quiz 1 or Quiz 2 the First Day of the Semester

	All Operations Correct	Some Operations Correct	No Operations Correct	Total
Arithmetic thinking (operations used)				
Figure with circle in square problem 1, quiz 1	45 63.4%	12 16.9%	14 19.7%	71 100.0%
Figure with trapezoid problem 4, quiz 2	36 54.6%	21 31.8%	9 13.6%	66 100.0%
Algebraic thinking (operations represented)				
Figure with circle in square problem 2, quiz 2	5 7.6%	9 13.6%	52 78.8%	66 100.0%
Figure with trapezoid problem 3, quiz 1	23 32.4%	10 14.1%	38 53.5%	71 100.0%

The responses to the arithmetic problems 1 and 4 demonstrate that most of the students answering these problems knew what operations to use to find each area. In contrast, the students responding to the similar algebraic problems 2 and 3 did not use those same operations in an algebraic fasharithmetic thought was required. Remarkably, in each algebraic problem, more than half the students did not express operations algebraically at all.

The circle-in-square problem shows the difference between arithmetic and algebraic performance most dramatically. The fact that 63.4 percent of the students responding to the arithmetic problem used operations correctly to evaluate the area indicates that they understood and could select the appropriate formulas for the mathematical relationships involved in this problem. But only 7.6 percent of those responding to the algebra problem demonstrated that they could correctly express the problem situation corresponding to these relationships when they had to build their own symbolic formula (and all who expressed it correctly solved it correctly, too). Only 21.2 percent of the responses to the algebraic form of the problem contained *any* correct use of algebraic notation at all. All those students who failed to use algebraic notation were not even on the right track.

The categorization of the responses for the trapezoid problem shows similar responses, but the distribution of students across the categories in the table is not so dramatic. However, the reason they are not so dramatic reveals further information about the students' affinity for an arithmetic approach to a solution.

One approach to problem 3 is to drop a perpendicular to the base and create a formula for the area of the rectangle on the left and the triangle on the right: $A(x) = 5x + (1/2)5(10 - x)$. Then set this equal to the given area, 32, and solve for x. Fourteen of the twenty-three students who got this problem right (of the seventy-one who tried) used this algebraic method. Nine used a more numerical approach. By completing the rectangle (fig. 7.3), they were able to deduce that the triangle had area $5(10) - 32 = 18$. Then they solved $A = (1/2)bh = (1/2)5h$ for h and subtracted the number h from 10 to find the desired side. This sequence avoids building a symbolic formula. Rather, students are able to work with known numbers except for the single step, which they solve using the well-known formula for the area of a triangle.

The responses to the set of paired problems with the same mathematical relationships show that many students with a great deal of training in algebra are still far more comfortable identifying and using operations on known numbers than they are expressing these same operations when the numbers are unknown. These responses can be interpreted as an indication of a difference between *being able to use operations* and *having operations as conceptual objects* that can be dealt with without reference to particular numbers. These two types of behavior characterize a difference between arithmetic and algebraic thought.

The high percentages of successful responses for the arithmetic form of each problem indicate that most of the students understood the quantitative situations described in words well enough to select the correct operations in the correct sequence when numbers could be used at each step. But the data also show that many of the students are not able to move beyond an arithmetic level of understanding to think algebraically about those same operations when the number is unknown.

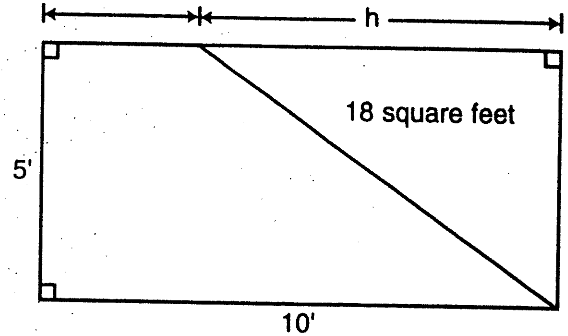

Fig. 7.3. Students' arithmetic approach to problem 3

Unknown and Dummy Variables

Students who perceive algebraic symbols as representations of specific numbers and combinations of symbols as directions for manipulating numbers will have difficulty perceiving algebraic objects at the level of operations and order. This section examines how a numerical or algebraic perception of the concept of *variable* affects students' abilities to think algebraically.

Love (1986, p. 49) defines algebraic thinking as

> not merely "giving meaning to symbols," but another level beyond that: concerning itself with those modes of thought that are essentially algebraic—for example, handling the as-yet-unknown, inverting and reversing operations, seeing the general in the particular. Becoming aware of these thought processes and in control of them, is what it means to think algebraically.

In this definition, being able to "handle" the as-yet-unknown really means being able to recognize, express, and manipulate operations. To function at this level, a student's perception of algebraic entities must shift from *number* to a focus on *operations* and *relationships* among numbers. Different kinds of variables are used to express these different kinds of thoughts.

Unknowns are variables (symbols, usually letters) used to represent *particular* numbers in thoughts about numbers. For example, the equation $3x + 2x = 20$ gives information about "x," a number. (The term *variable* may be unfortunate, since some variables, such as unknowns, are not intended to "vary" [Schoenfeld and Arcavi 1988]. In contrast, dummy variables, also known as placeholders, are letters used to hold places where *any* number could be used in thoughts about operations and order (Usiskin 1988; Esty forthcoming). For example, the identity $3x + 2x = 5x$ gives information about the operations of multiplication and addition. This identity gives no information about x, which is merely a placeholder.

The conception of algebra as being *about* operations and order is required in order to appreciate the way that written mathematics employs dummy variables to express methods using identities, theorems, and formulas (Esty 1993; Esty forthcoming). Identities express alternative sequences of operations using dummy variables. How do we subtract a negative number from a positive number? The identity $a - (-b) = a + b$ expresses a method for evaluating $5 - (-3)$. It holds for all values of a and b. The symbols a and b hold places, but not particular values.

Theorems that express the equivalence of equations use dummy variables. The following is an example: $10^a = b$ is equivalent to $a = \log b$. The purpose of a and b in this theorem is to hold places so that the relationship between the operations of exponentiation and taking logarithms can be expressed. The symbol a could be replaced in both places by c or x without affecting the meaning.

Reconsider the algebraic version of the problem with a garden outside a circle in a square (problem 2, fig. 7.1). The key formula $A(x) = x^2 - \pi(x/2)^2$ holds for any side and contains no information about the numerical value of x, the side. The subsequent equation employs x as an unknown, but the numerical information it contains is not relevant until the final, computational, step in the solution process.

The use of letters as dummy variables is a characteristic of algebraic thought. Algebraic concepts and symbolic algebraic language reinforce each other. Without the language, such concepts are difficult to articulate. At the same time, the existence of these concepts creates a need to develop a language appropriate for their study.

CONCLUDING COMMENTS

The students' responses to the action-research study illustrate key differences between arithmetic and algebraic thinking. These differences affect students' abilities to use symbolic notation to think about operations and order. However, the process of moving from arithmetic to algebraic thinking is nontrivial (Sfard and Linchevski 1994). Most precalculus students in the study did not represent operations when it was appropriate to do so. This evidence is particularly striking because it suggests that the students did not grasp the very purpose of the algebraic notation, in spite of using the notation daily in two or more years of high school algebra.

The data from the parallel problem sets also indicate the potential for a serious problem with mismatched instruction. Students who are at a numerical level of thought will not necessarily be prepared to work with algebraic objects of thought at the conceptual level assumed by algebra texts and instructors.

The symbolic language of algebra takes on meaning as students study the ways in which it extends their ability to think about mathematical concepts. By studying word problems, students encounter examples in which algebraic language and the use of dummy variables enable them to think in new, more abstract ways about the conceptual objects of algebra: operations and order.

"Algebra is a symbol system of unparalleled power for communicating quantitative information and relationships" (Fey 1989, p. 207). If students are to be given access to this power, then it is important that we understand and make explicit to students the underlying concepts of the discipline. Symbolic notation is essential for conceptualizing algebra.

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